UNIT-2

LINEAR TRANSFORMATION AND DIAGONALIZATION

Sec 1

Linear Transformation :-

Defn

Let V and W be Vector Spaces (over F). Then a function $T: V \to W$ is said to be linear transformation from V to W, if $\forall\ x, y \in V$ and $\alpha \in F$, we have,

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$.

It is also called as Linear.

Examples:-

1) Trivial L.T

$T: V \to W \ni T(v) = 0 ; v \in V$

2) Identity L.T

$T: V \to V \ni T(v) = v ; v \in V$

3) Linear Transformation

$T: V_3(R) \to V_3(R)$

$\ni T(a, b, c) = (a, 0, 0)$

4) L.T (reflection about x-axis)

$T: V_2(R) \to V_2(R)$

$\ni T(a_1, a_2) = (a_1, -a_2)$.

5) L.T (Projection on x-axis)

$T: V_2(R) \to V_2(R)$

$\ni T(a_1, a_2) = (a_1, 0)$.

6) $T : M_{m\times n}(F) \longrightarrow M_{m\times n}(F)$

def by $T(A) = A^T$

Problems:-

1) S.T $T : V_2(R) \to V_2(R)$ def by $T(a,b) = (2a-3b, a+4b)$ is a linear transformation.

Soln

Let $x = (a,b)$; $y = (c,d)$; and $\alpha \in R$

(i) $T(x+y) = T((a,b) + (c,d))$

$= T(a+c, b+d)$

$= (2(a+c) - 3(b+d), a+c+4(b+d))$

$= (2a+2c-3b-3d, a+c+4b+4d)$

$= (2a-3b+2c-3d, a+4b+c+4d)$

$= (2a-3b, a+4b) + (2c-3d, c+4d)$

$= T(a,b) + T(c,d)$

$= T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha(a,b))$

$= T(\alpha a, \alpha b)$

$= (2\alpha a - 3\alpha b, \alpha a + 4\alpha b)$

$= \alpha(2a-3b, a+4b)$

$= \alpha T(a,b)$

$= \alpha T(x)$.

$\therefore$ T is a linear transformation.

2) Let $C[-1,1]$ be the V.S of all Continuous fns on the interval $[-1,1]$. A map T is defined by $T(f(x)) = x f'(x)$ for all $f(x) \in C[-1,1]$. S.T, T is a linear map.

Soln

Let $f(x), g(x) \in C[-1,1]$ and $\alpha \in F$

def by $T(f(x)) = x f'(x)$

(i) $T(f(x)+g(x)) = x\,(f+g)'(x)$

$= x\,(f'(x) + g'(x))$

$= x f'(x) + x g'(x)$

$= T(f(x)) + T(g(x))$

(ii) $T(\alpha f(x)) = x\,[\alpha f(x)]'$

$= x\,\alpha f'(x)$

$= \alpha\, x f'(x)$

$= \alpha T(f(x))$

$\therefore$ T is linear

3) Check $T: \mathbb{R} \to \mathbb{R}$ by $T(x) = x^2$ is a linear map.

Soln

(i) $T(x+y) = (x+y)^2$

$= x^2 + y^2 + 2xy$

$= T(x) + T(y) + 2xy$

$\neq T(x) + T(y)$

$\therefore$ T is not linear.

4) $T: \mathbb{R}^2 \to \mathbb{R}^2$
by $T(x,y) = (x, y+3)$
Check whether T is linear.

Soln

Let $x = (x_1, x_2)$
then $T(x) = T(x_1, x_2) = (x_1, x_2+3)$

and $y = (y_1, y_2)$
then $T(y) = T(y_1, y_2) = (y_1, y_2+3)$

(i) $T(x+y) = T((x_1,x_2) + (y_1, y_2))$
$= T(x_1+y_1, x_2+y_2)$
$= (x_1+y_1, x_2+y_2+3)$
$= (x_1, x_2) + (y_1, y_2+3)$
$\neq T(x) + T(y)$

$\therefore$ T is not linear.

5) Let $T: V_3(R) \to V_2(R)$ by $T(a_1, a_2, a_3) = (a_1 - a_2, a_3)$.
Check T is linear.

Soln

Let $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$
and $\alpha \in F$

(i) $T(x+y) = T((a_1,a_2,a_3) + (b_1, b_2, b_3))$
$= T(a_1+b_1, a_2+b_2, a_3+b_3)$
$= (a_1+b_1-a_2-b_2, a_3+b_3)$
$= (a_1-a_2+b_1-b_2, a_3+b_3)$
$= (a_1-a_2, a_3) + (b_1-b_2, b_3)$

$$= T(a_1,a_2,a_3) + T(b_1,b_2,b_3)$$

$$= T(x)+T(y)$$

(ii) $T(\alpha x) = T(\alpha(a_1,a_2,a_3))$

$$= T(\alpha a_1, \alpha a_2, \alpha a_3)$$

$$= (\alpha a_1 - \alpha a_2, \alpha a_3)$$

$$= \alpha(a_1-a_2,a_3).$$

$$= \alpha T(a_1,a_2,a_3)$$

$$= \alpha T(x)$$

$\therefore$ T is linear.

H.W

1) Let $T: V_3(R) \to V_2(R)$ defined by $T(x_1,x_2,x_3) = (x_1-x_2, x_2+x_3)$ $\forall (x_1,x_2,x_3) \in V_3(R)$. P.T T is linear.

2) Define $T: V_2(R) \to V_2(R)$ by $T(a_1, a_2) = (2a_1+a_2, a_1)$ S.T T is linear

Sec 2

Null Space and Ranges :-

Defn :- Null Space

Let V and W be Vector Space and let $T: V \to W$ be a linear transformation. Then the set of all $x \in V$ $\ni T(x) = 0$ is called Null Space of T or Kernel of T and it is denoted by $N(T)$ or $Ker(T)$

(i.e) $N(T)$ (or) $Ker(T) = \{x / x \in V$ and $T(x) = 0\}$.

and the dimension of Null Space of T is called Nullity of T denoted by Nullity(T).

(i.e) Nullity (T) = Dimension of $N(T)$

Range :-

~~Let V and W be a finite dimensional V.S and let T: V → W be a linear transformation. Then dimension of Null Space of T is called Nullity of~~

Let V and W be Vector spaces and let $T: V \to W$ be a linear transformation. Then the subset of W consisting of all images under T of Vector in V is called range or image of T and it is denoted by $R(T)$

$$R(T) = \{w \in W / T(v) = w \text{ for some } v \in V\}.$$

and the dimension of $R(T)$ is called rank of T denoted by Rank(T).

(i.e) Rank(T) = Dim of $R(T)$

Example

1) $T: V \to W$

def $T(v) = 0 \quad \forall\ v \in V$

(a) $N(T) = \{x / T(x) = 0\}$

$\Rightarrow N(T) = V$

(b) Nullity of T = dim of $N(T)$
= dim of V.

(c) $R(T) = \{ w \in W / T(v) = w \}$

here $T(v) = 0$.

$\therefore R(T) = \{0\}$.

(d) Rank of T = dim of $R(T)$
$= 0$.

2) $T: V \to V$
def by $T(v) = v$

then, (a) $N(T) = \{ x / T(x) = 0 \}$.
but here $T(x) = x$
$\Rightarrow N(T) = \{0\}$.

(b) Nullity(T) = Dim of $N(T)$
$= 0$

(c) $R(T) = \{ w \in W / T(v) = w \}$
but here $T(v) = v \in V$
$\Rightarrow R(T) = V$

(d) Rank of T = dim of $R(T)$.
= dim of V.

<u>Theorem</u>

Let V and W be Vector Spaces and $T: V \to W$ be linear transformation. Then
(a) Null space of T (ie) $N(T)$ is a subspace of V
(b) Range T (ie) $R(T)$ is a Subspace of W.

Proof

(a) We know that

$$N(T) = \{x / x \in V \text{ and } T(x) = 0\}$$

$\because T(0) = 0$

We have $0 \in N(T)$

To prove $N(T)$ is a subspace of V

we have to prove

(i) for $x, y \in N(T) \Rightarrow x+y \in N(T)$

(ii) for $x \in N(T) \Rightarrow cx \in N(T)$

(i) If $x, y \in N(T)$

then $T(x) = 0$ & $T(y) = 0$. — (1)

consider,

$$T(x+y) = T(x) + T(y) \quad (\because T \text{ is a L.T})$$

$$= 0 + 0. \ (\text{by } (1))$$

$$= 0.$$

$\Rightarrow x+y \in N(T)$

(ii) If $x \in N(T)$

then $T(x) = 0$ — (2)

consider

$$T(cx) = cT(x) \quad (\because T \text{ is a L.T})$$

$$= c(0)$$

$$= 0.$$

$\Rightarrow cx \in N(T)$.

$\therefore$ $N(T)$ is a subspace of T.

(b) We know that

$$R(T) = \{w \in W / T(v) = w \text{ for some } v \in V\}.$$

$\because T(0) = 0$

We have $0 \in R(T)$

To prove $R(T)$ is a subspace of W

We have to prove.

(i) for $x, y \in R(T)$ then $x+y \in R(T)$

(ii) for $c \in F$ & $x \in R(T)$

then $cx \in R(T)$

(i) If $x, y \in R(T)$

then $T(u) = x$ for $u \in V$

and $T(v) = y$ for $v \in V$.

consider,

~~T(x+y) = T(x)+T(y) (∵ T is a L.T)~~

$T(u+v) = T(u) + T(v)$ ($\because T$ is a L.T)

$= x + y$

$\Rightarrow x+y \in R(T)$

(ii) If $x \in R(T)$ & $c \in F$.

then $T(u) = x$ for some $u \in V$

Consider.

$T(cu) = cT(u)$ ($\because T$ is a L.T).

$= cx$.

$\Rightarrow cx \in R(T)$

$\therefore$ $R(T)$ is a subspace of W

Problems :-

1) Let $T: V_2(R) \to V_3(R)$ be a linear map defined by $T(x_1, x_2) = (x_1 - x_2, 0, 0)$. Find the null space $N(T)$ and range T, and hence Nullity of T and Rank T

Soln

Null Space $= \{(x_1, x_2) / T(x_1, x_2) = 0\}$

$= \{(x_1, x_2) / (x_1 - x_2, 0, 0) = 0\}$

$= \{(x_1, x_2) / x_1 - x_2 = 0\}$

$= \{(x_1, x_2) / x_1 = x_2\}$

$= \{(x_1, x_1) / x_1$ is any real number$\}$

$\therefore \quad N(T) = (1, 1)$

Nullity of $T =$ Dim of $N(T)$

$= 1$

Range of $T = \{T(x_1, x_2) / x_1$ and x_2 are any real nos$\}$

$= \{(x_1 - x_2, 0, 0) / x_1, x_2 \in R\}$

$= \{(x_1, 0, 0), (-x_2, 0, 0) / x_1, x_2 \in R\}$.

$R(T) = \{(1, 0, 0), (-1, 0, 0)\}$.

In the above two vectors, the one can be written as the linear combination of the other.

$\therefore \quad R(T) = \{(1, 0, 0)\}$.

Rank $(T) = 1$

2) Let a linear map $T: P_3(R) \to P_2(R)$ be defined by $T(f(x)) = f'(x)$. Find $N(T)$ and $R(T)$ and also nullity T and rank T.

Soln

Let $f(x) \in P_3(R) \Rightarrow f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Null Space $= \{f(x) / T(f(x)) = 0\}$.

Given $T(f(x)) = f'(x)$

$= a_1 + 2a_2 x + 3a_3 x^2$

$\Rightarrow a_1 = a_2 = a_3 = 0$.

$\therefore$ Null Space $= \{a_0 / a_0 \in R\}$.

$= \{1 /$ for any Value of $a_0\}$

$\Rightarrow N(T) = \{1\} =$ set of all Polynomial of degree 0.

Nullity $(T) =$ dim of $N(T)$

$= 1$

Range of $T = \{T(f(x)) / f(x) \in P_3(R)\}$

$= \{f'(x) / f(x) \in P_3(R)\}$

$= \{a_1 + 2a_2x + 3a_3x^2 / a_1, a_2, a_3 \in R\}$.

$\Rightarrow R(T) = P_2(R)$.

Rank T $=$ dim of $P_2(R)$.

3) Let $T: V_3 \rightarrow V_3(R)$ be a linear map defined by $T(x_1, x_2, x_3) = (x_1 - x_2, 2x_2 + x_3, 0)$. find $N(T)$ and $R(T)$. Also find nullity T and Rank T.

<u>Soln</u>

Null Space $= \{(x_1, x_2, x_3) / T(x) = 0\}$.

Here,

$T(x) = (x_1 - x_2, 2x_2 + x_3, 0) = (0, 0, 0)$

$$\begin{array}{l|l} \Rightarrow x_1 - x_2 = 0 & 2x_2 + x_3 = 0. \\ x_1 = x_2 & -2x_2 = x_3. \end{array}$$

$\Rightarrow N(T) = \{(x_2, x_2, -2x_2) / x_2 \in R\}$.

$= \{(1, 1, -2) /$ for any $x_2 \in R\}$.

$\therefore$ dim $N(T) = 1$.

$$R(T) = \{ T(x_1, x_2, x_3) \,/\, (x_1, x_2, x_3) \in V \}$$

$$= \{ (x_1 - x_2,\ 2x_2 + x_3,\ 0) \,/\, (x_1, x_2, x_3) \in V \}$$

$$= \{ (x_1 - x_2 + 0x_3,\ 0x_1 + 2x_2 + x_3,\ 0x_1 + 0x_2 + 0x_3) \,/\, (x_1, x_2, x_3) \in V \}$$

$$= \{ (x_1, 0, 0) + (-x_2, 2x_2, 0) + (0x_3, x_3, 0x_3) \,/\, (x_1, x_2, x_3) \in V \}$$

$$= \{ (1,0,0),\ (-1,2,0),\ (0,1,0) \}.$$

In the above vectors $(-1,2,0)$ can be a linear combination of $(1,0,0)$ & $(0,1,0)$

$\therefore\ R(T) = \{(1,0,0)\ \&\ (0,1,0)\}.$

Rank of T = dim of $R(T)$

$= 2.$

<u>H.W</u>

1) Let $T: V_3(R) \to V_2(R)$ be the L.T def by
$T(a_1, a_2, a_3) = (a_1 - a_2,\ 2a_3)$. Find $N(T)$ & $R(T)$.

<u>Ans</u> $N(T) = \{1,1,0\}$; Nullity $= 1$; $R(T) = \{(1,0), (0,2)\}$.
Rank of $T = 2$.

Sec 3

Dimension Theorem (Rank- Nullity Theorem)

Theorem 1

Statement

Let V and W be Vector Spaces and let $T: V \to W$ be a linear transformation. If V is finite dimensional, then

$$\text{Nullity}(T) + \text{Rank}(T) = \text{Dim}\, V.$$

(or)

$$\text{Dim}(N(T)) + \text{Dim}(R(T)) = \text{Dim}\, V.$$

Proof

Let $\{u_1, u_2, \ldots, u_k\}$ be a basis of Null Space $N(T)$.

$\because$ $N(T)$ is a subspace of V,

the basis $\{u_1, u_2, \ldots, u_k\}$ of $N(T)$ can be extended to basis of V, say $\{u_1, u_2, u_3, \ldots, u_k, v_1, v_2, \ldots, v_n\}$.

(i) To prove,

$\{Tv_1, Tv_2, \ldots, Tv_n\}$ is a basis of $R(T)$.

Let $w \in R(T)$. Then $\exists\, v \in V$

$\ni T(v) = w$.

Now,

$$v = \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{n} \beta_j v_j$$

$$\therefore \quad w = T(v) = T\left(\sum \alpha_i u_i + \sum \beta_j v_j\right)$$

$$= \sum \alpha_i T(u_i) + \sum \beta_j T(v_j)$$

$\because$ $\{u_1, u_2, \ldots, u_k\}$ is a basis of $N(T)$

By defn $T(u_i) = 0$.

$$\therefore \quad w = T(v) = \sum_{j=1}^{n} \beta_j T(v_j)$$

$\Rightarrow$ $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ spans $R(T)$.

~~$\therefore$ $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis of $R(T)$~~

(ii) We claim that $\{T(V_j)\}$ is linearly independent,

Consider the linear Combination,

$$\sum_{j=1}^{n} \beta_j T(V_j) = 0.$$

$$\Rightarrow T\left(\sum_{j=1}^{n} \beta_j V_j\right) = 0 \quad (\because T \text{ is linear}.$$

$$\Rightarrow \sum_{j=1}^{n} \beta_j V_j \in N(T)$$

But $\{u_1, u_2, \dots, u_K\}$ is the basis of $N(T)$.

$$\therefore \sum_{j=1}^{n} \beta_j V_j = \sum_{i=1}^{K} \alpha_i u_i$$

$$\therefore \sum_{j=1}^{n} \beta_j V_j - \sum_{i=1}^{K} \alpha_i u_i = 0.$$

$\because \{u_1, u_2, \dots, u_K, V_1, V_2, \dots, V_n\}$ is a basis of V

We have $\alpha_i = 0$ & $\beta_j = 0$.

$\therefore \{T(V_j)\}$ is Linearly independent.

$\Rightarrow \{T(V_1), T(V_2), \dots, T(V_n)\}$ is a basis of $R(T)$

$\text{Dim}(R(T)) = n$.

(ie) Rank $T = n$

(add and sub K.

$$\Rightarrow \text{Rank}T = n + K - K.$$

$$= \dim V - \dim N(T).$$

$$= \dim V - \text{Nullity } T$$

$$\Rightarrow \dim V = \text{Rank } T + \text{Nullity } T$$

Theorem 2

Let V and W be Vector Spaces and let $T: V \to W$ be linear transformation. Then T is 1-1 iff $N(T) = 0$.

Soln

Suppose T is 1-1.

$\Rightarrow T(x) = T(y)$

then, $x = y$ —— (1) $\forall\ x, y \in V$.

Let $x \in N(T)$.

$\Rightarrow T(x) = 0 = T(0)$.

$\Rightarrow T(x) = T(0)$

by (1).

$x = 0$.

Hence $N(T) = \{0\}$

Conversely, $N(T) = \{0\}$.

Let $T(x) = T(y)$

We have to prove $x = y$

Sothat, T is 1-1.

$T(x) = T(y)$.

$\Rightarrow T(x) - T(y) = 0$.

$\Rightarrow T(x - y) = 0$.

$\Rightarrow x - y \in N(T) = \{0\}$.

$\Rightarrow x - y = 0$.

$\Rightarrow x = y$

$\therefore$ T is 1-1.

Theorem 3 :-

Let V and W be V.S of equal dimension and let $T: V \to W$ be linear. Then the following are equivalent

(a) T is 1-1

(b) T is onto

(c) $\text{Rank}(T) = \dim(V)$.

<u>Proof</u>

(a) $\Rightarrow$ (b)

T is 1-1.

by thm (2)

$N(T) = \{0\}$.

$\Rightarrow$ Nullity $T = 0$.

by thm (1)

$\dim(V) = \text{Rank}\,T + \text{Nullity}\,T$

$\Rightarrow \dim V = \text{Rank}\,T$

$\Rightarrow \dim V = \dim(R(T))$

$\Rightarrow V = R(T)$.

$\Rightarrow$ T is onto.

(b) $\Rightarrow$ (c).

T is onto.

$R(T) = V$.

$\dim R(T) = \dim V$

$\text{Rank}\,T = \dim V$.

(c) $\Rightarrow$ (a).

$\text{Rank}\,T = \dim V$ — (1)

W.K.T by Thm 1

$\dim V = \text{Rank}\,T + \text{Nullity}\,T$ — (2)

by (1) & (2)

Nullity $T = 0$.

$\Rightarrow \dim N(T) = 0$.

by thm (2), T is 1-1.

$\therefore$ (a), (b) and (c) are equivalent

Problems.

1) Let $T: P_2(R) \to P_3(R)$ be the linear transformation defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(x)\,dx$.

check 1. T is 1-1 or not

2. T is onto or not?

Soln

W.K.T $\{1, x, x^2\}$ is a basis of $P_2(R)$.

We have to find

$$R(T) = \text{Span}\{T(1), T(x), T(x^2)\}.$$

$$T(1) = 2(0) + \int_0^x 3(1)\,dx = 3x\Big]_0^x = 3x.$$

$$T(x) = 2(1) + \int_0^x 3(x)\,dx = 2 + \frac{3x^2}{2}\Big]_0^x = 2 + 3x^2/2$$

$$T(x^2) = 2(2x) + \int_0^x 3(x^2)\,dx = 4x + \frac{\cancel{3}x^3}{\cancel{3}}\Big]_0^x = 4x + x^3.$$

$$\therefore R(T) = \text{Span}\left\{3x, 2 + \frac{3x^2}{2}, 4x + x^3\right\}.$$

Rank T = 3.

But $\dim P_3(R) = 4$.

T is not (onto).

$\because$ Rank $T \neq \dim P_3(R)$

from the Dimension theorem,

Nullity T + Rank T = dim(V).

Nullity T + 3 = $\dim P_2(R)$.

$= 3$.

$\Rightarrow$ Nullity T = 0

(i.e) $N(T) = \{0\}$

by theorem 2, T is 1-1.

Example 2:-

Let $T: V_2(R) \rightarrow V_2(R)$ be the linear transformation defined by $T(a_1,a_2) = (a_1+a_2, a_1)$.

Check T is 1-1 and onto or not?

Soln

$$N(T) = \{(a_1,a_2) \,/\, T((a_1,a_2)) = 0\}$$

$$= \{(a_1,a_2) \,/\, (a_1+a_2, a_1) = 0\}$$

$$\Rightarrow a_1 = 0 \ \& \ a_1+a_2 = 0.$$

$$a_2 = 0.$$

$$= \{0\}.$$

Nullity $T = 0$.

$\therefore$ by theorem 2, T is 1-1.

~~by theorem~~

$$R(T) = \{T(a_1,a_2) \,/\, a_1, a_2 \in R\}.$$

$$= \{(a_1+a_2, a_1) \,/\, a_1 a_2 \in R\}.$$

$$= \{(a_1+a_2, a_1+0a_2) \,/\, a_1, a_2 \in R\}.$$

$$= \{(a_1, a_1) + (a_2, 0a_2) \,/\, a_1, a_2 \in R\}$$

$R(T) = \{(1,1), (1,0)$ / for any $a_1, a_2 \in R\}$.

Rank $T = 2$.

$\therefore$ Rank $T = \dim V = \dim V_2(R)$ (Here).

$\therefore$ by theorem 3,

T is onto.

3) Let $T: V_3(R) \rightarrow V_3(R)$ be the linear transformation defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$, find a basis and dimension $R(T)$ and $N(T)$ (i.e) rank T and nullity.

Soln

Let $S = (x+2y-z, y+z, x+y-2z)$.

To prove a basis

(i) We have to prove S is L.I

(ii) $L(S) = V_3(R)$

(i) Consider the linear combination

$a_1(x+2y-z) + a_2(y+z) + a_3(x+y-2z) = 0$.

equating the coeff of x, y, z

then $a_1 + a_3 = 0$. — ①

$2a_1 + a_2 + a_3 = 0$. — ②

$-a_1 + a_2 - 2a_3 = 0$ — ③.

$1 \times ② \Rightarrow 2a_1 + \cancel{a_2} + a_3 = 0$.

$-1 \times ③ \Rightarrow a_1 - \cancel{a_2} + 2a_3 = 0$

$3a_1 + 3a_3 = 0$. — ④.

No need.

solve ① & ④ we get

$a_1 = 0, a_2 = 0$.

$\therefore$ S is L.I.

(ii) $L(S) = \{ a_1(x+2y-z) + a_2(y+z) + a_3(x+y-2z) \,/\, a_1, a_2, a_3 \in R \}$

$= \{ (a_1+a_3)x + (2a_1 + a_2 + a_3)y + (-a_1 + a_2 - 2a_3)z \,/\, a_1, a_2, a_3 \in R \}$.

$= \{$ for any $(a,b,c) \in V_3(R)$

$(a,b,c) = (a_1+a_2)x + (2a_1 + a_2 + a_3)y + (-a_1 + a_2 - 2a_3)z$.

has a unique Linear combination.

$\therefore L(S) = V_3(R)$

$\Rightarrow L(S)$ is a basis of $V_3(R)$.

Null Space = $N(T) = \{(x, y, z) / T(x,y,z) = 0\}$.

$\Rightarrow \quad x + 2y - z = 0.$

$y + z = 0.$

$x + y - 2z = 0.$

$$\begin{matrix} 2 & -1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{matrix}$$

$$\frac{x_1}{2+1} = \frac{x_2}{0-1} = \frac{x_3}{1} \Rightarrow \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

$N(T) = \{(3,-1,1) / T(x,y,z) = 0\}$.

Nullity $T = 1$. It is linearly dependent. $\therefore$ No basis

Range of $T = \{T(x,y,z) / x,y,z \in R\}$.

$= \{(x+2y-z,\ 0x+y+z,\ x+y-2z) / x,y,z \in R\}$

$= \{(x, 0x, x) + (2y, y, y) + (-z, z, -2z)\}$.

$= \{(1,0,1) , (2,1,1) , (-1,1,-2) / \text{for any } x,y,z \in R\}$.

$(-1,1,-2)$ can be written as the linear combination of $(1,0,1)$ and $(2,1,1)$. (Any one can be written as the linear combination of other 2).

$R(T) = \{(1,0,1), (2,1,1)\}$.

Rank $T = \dim R(T)$.

$= 2.$

To check the linearly independent of $R(T)$

Consider the linear combination

$a_1(1,0,1) + a_2(2,1,1) = 0$

$$a_1 + 2a_2 = 0.$$

$$a_2 = 0.$$

$$a_1 + a_2 = 0.$$

$$\Rightarrow a_1 = 0, a_2 = 0.$$

$\therefore$ The Vectors are linearly independent and It forms the basis of $R(T)$

4) Let $T: V_2(R) \longrightarrow V_3(R)$ be a mapping defined by $T(a,b) = (a+b, a-b, b)$. S.T, T is a linear transformation from $V_2(R)$ to $V_3(R)$. Find $N(T)$ and $R(T)$, Nullity T and Rank(T).

Soln

Let $x = (a,b) \in V_2(R)$

$\Rightarrow T(a,b) = (a+b, a-b, b)$

and $y = (c,d) \in V_2(R)$

$\Rightarrow T(c,d) = (c+d, c-d, d)$

(i) $T(x+y) = T((a,b) + (c,d))$

$= T(a+c, b+d)$

$= (a+c+b+d, a+c-b-d, b+d)$

$= (a+b+c+d, a-b+c-d, b+d)$

$= (a+b, a-b, b) + (c+d, c-d, d)$

$= T(a,b) + T(c,d)$

$= T(x) + T(y)$

(ii) $T(cx) = T(c(a,b))$

$= T(ca, cb) = (ca+cb, ca-cb, cb)$

$= c(a+b, a-b, b) = cT(a,b) = cT(x)$

$\therefore$ T is Linear.

$N(T) = \{(a,b) / T(a,b) = 0\}$.

$\Rightarrow (a+b, a-b, b) = 0$.

$b = 0, \ a-b = 0$.

$a = b$.

$\Rightarrow a = 0$.

$N(T) = \{(0,0)\}$.

Nullity $T = 0$.

Range of $T = R(T) = \{T(a,b) / a,b \in R\}$.

$= \{(a+b, a-b, b) / a,b \in R\}$.

$= \{(a+b, a-b, 0a+b) / a,b \in R\}$

$= \{(a,a,0a) + (b,-b,b) / a,b \in R\}$

$= \{(1,1,0), (1,-1,1) / a,b \in R\}$.

$\therefore \dim R(T) = \text{Rank } T = 2$.

Sec4

Matrix Representation of a Linear Transformation:-

Ordered Basis:-

Let V be a finite dimensional Vector Space. An Ordered basis for V is a basis for V assigned with a Specific order.

(i.e) An ordered basis for V is a finite Sequence of linearly independent Vectors in V that generates V.

(Eg):-

(i) In F^3, $\{e_1, e_2, e_3\}$ can be considered an Ordered basis.

(ii) In F^n, $\{e_1, e_2, \ldots, e_n\}$ is the Standard Ordered basis

(iii) In $P_n(F)$, $\{1, x, x^2, \ldots, x^n\}$ is the Standard ordered Basis.

Matrix Representation:-

The $m\times n$ matrix which we have associated with a linear transformation $T: V \to W$ depends on the choice of the basis for V and W.

Co-ordinate Vector

Let $\beta = \{u_1, u_2, \ldots u_n\}$ be an ordered basis for a finite dimensional Vector space V.

for $x \in V$, let $a_1, a_2, \ldots, a_n$ be the Unique Scalar $\ni$ $x = \sum_{i=1}^{n} a_i u_i$

Now, we define the Co-ordinate Vector of x relative to β, denoted by

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Matrix Representation :-

We call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$, the matrix representation of T in the ordered basis β and γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$.

then, we write $A = [T]_\beta$.

Theorem :-

Let V and W be finite-dimensional vector space with ordered bases β and γ respectively and let $T_1, T_2 : V \to W$ be linear transformation. Then,

(a) $[T_1 + T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$

(b) $[cT_1]_\beta^\gamma = c[T_1]_\beta^\gamma$ for all scalars c.

Proof

(a) Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$.

$\exists$ unique scalars a_{ij} and b_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$

$$\ni T_1(v_j) = \sum_{i=1}^{m} a_{ij} w_i \text{ and}$$

$$T_2(v_j) = \sum_{i=1}^{m} b_{ij} w_i \text{ for } 1 \leq j \leq n$$

Hence $(T_1 + T_2)(v_j) = \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_i$

Thus $\left([T_1 + T_2]_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$

(b) $cT(v_j) = \sum_{i=1}^{n} c a_{ij} w_i$

$[cT]_\beta^\gamma = c a_{ij} = c[T]_\beta^\gamma$.

Problems

1) Obtain the matrix representing the linear transformation $T: V_3(R) \to V_3(R)$ gn by $T(a,b,c) = (3a, a-b, 2a+b+c)$ w.r to standard basis.

Soln

Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis of $V_3(R)$

(for image)

then, $T(e_1) = T(1,0,0) = (3a, a, 2a) = (3,1,2) = 3e_1 + e_2 + 2e_3$

$T(e_2) = T(0,1,0) = (0,-b,b) = (0,-1,1) = 0e_1 - e_2 + e_3$

$T(e_3) = T(0,0,1) = (0,0,c) = (0,0,1) = 0e_1 + 0e_2 + e_3$

Thus, the matrix representing T,

$$[T]_\beta = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

2) Let $T: V_2(R) \to V_3(R)$ be the L.T def by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let $\beta = \{e_1, e_2\}$ and $\gamma = \{e_1, e_2, e_3\}$ be the standard basis, w.r to $V_2(R)$ and $V_3(R)$ respectively. Find the corresponding matrix.

Soln

Let $\beta = \{e_1, e_2\}$ be the standard basis of $V_2(R)$

then $T(e_1) = T(1,0) = (a_1, 0, 2a_1) = (1, 0, 2) = e_1 + 0e_2 + 2e_3$

$T(e_2) = T(0,1) = (3a_2, 0, -4a_2) = (3, 0, -4) = 3e_1 + 0e_2 - 4e_3$.

Thus the corresponding matrix is

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

3) obtain the matrices for the linear transformation $T: V_2(R) \to V_2(R)$ gn by $T(a,b) = (-b,a)$, with respect to the

(a) standard basis

(b) the basis $\{(1,2), (1,-1)\}$ for both domain and range.

Soln

(a) Let $B = \{e_1, e_2\}$ be the standard basis for $V_2(R)$.

$$T(e_1) = T(1,0) = (0,1) = 0e_1 + e_2$$

$$T(e_2) = T(0,1) = (-1,0) = -e_1 + 0e_2$$

The corresponding matrix is $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) Let $W = \{(1,2), (1,-1)\}$ be the basis for $V_2(R)$.

and $T(a,b) = (-b,a)$

then, $T(1,2) = (-2,1)$

$T(1,-1) = (1,1)$

Domain matrix $\} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$

then $\exists$ a_1 and a_2

$\ni (-2,1) = a_1(1,2) + a_2(1,-1)$

$$-2 = a_1 + \cancel{a_2}$$
$$1 = 2a_1 - \cancel{a_2}$$
$$-1 = 3a_1$$
$$\boxed{a_1 = -1/3}$$

$$-2 = -1/3 + a_2$$
$$a_2 = -2 + 1/3 = -5/3$$

$\therefore$ Column matrix of $(-2,1) = \begin{bmatrix} -1/3 \\ -5/3 \end{bmatrix}$

and $(1,1) = a_1(1,2) + a_2(1,-1)$

$$1 = a_1 + \cancel{a_2}$$
$$1 = 2a_1 - \cancel{a_2}$$
$$3a_1 = 2$$
$$a_1 = 2/3$$

$$1 = 2/3 + a_2$$
$$a_2 = 1 - 2/3 = 1/3.$$

$\therefore$ column matrix of $(1,1) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

(Range matrix).

$\therefore$ The required matrix is $\begin{pmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{pmatrix}$.

H.W

1) Let $T: V_3(R) \to V_2(R)$ gn by $T(a,b,c) = (a+b, 2c-a)$.
find the Corresponding matrix with respect to

(a) Standard basis and

(b) $\{(1,0,-1), (1,1,1), (1,0,0)\}$ as a basis for $V_3(R)$ and $\{(0,1), (1,0)\}$ for a $V_2(R)$

(c) $T: V_2(R) \to V_3(R)$ $T(a_1, a_2) = (2a_1 - a_2, 3a + 4a_2, a_1)$.

Ans

(a) $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{pmatrix}$

but γ is (e_2, e_1)

$\begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$

4) Let $T: P_3(R) \longrightarrow P_2(R)$ be the linear transformation def by $T(f(x)) = f'(x)$. Let β and γ be the standard Ordered Basis for $P_3(R)$ and $P_2(R)$ respectively. Find the Corresponding matrix.

Soln

Let $\beta = \{1, x, x^2, x^3\}$ be the basis for $P_3(R)$

and $\gamma = \{1, x, x^2\}$ be the basis for $P_2(R)$.

$T(1) = \frac{d}{dx}(1) = 0 = 0(1) + 0(x) + 0(x^2) = 0$

$T(x) = 1 = 1(1) + 0(x) + 0(x^2) = 1$

$T(x^2) = 2x = 0(1) + 2(x) + 0(x^2)$

$T(x^3) = (3x^2) = 0(0) + 0(x) + 3(x^2)$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

5) Let $T_1 : V_2(R) \longrightarrow V_3(R)$ and $T_2 : V_2(R) \longrightarrow V_3(R)$ be the linear transformation respectively defined by $T_1(a_1, a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$ and $T_2(a_1,a_2) = (a_1-a_2, 2a_1, 3a_1+2a_2)$.
Let β and γ be the standard bases of $V_2(R)$ and $V_3(R)$ respectively. Verify

$$[T_1+T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma.$$

Soln

Let $\beta = (e_1, e_2)$ and $\gamma = (e_1, e_2, e_3)$ be the Standard basis of $V_2(R)$ and $V_3(R)$ respectively.

Given $T_1(a_1,a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$

$$T_1(e_1) = (a_1, 0, 2a_1) = (1,0,2) = 1e_1+0e_2+2e_3)$$

$$T(e_2) = (3a_2, 0, -4a_2) = (3,0,-4) = 3e_1+0e_2-4e_3.$$

$$[T_1]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$$

Given $T_2(a_1,a_2) = (a_1-a_2, 2a_1, 3a_1+2a_2)$

$$T_2(e_1) = (a_1, 2a_1, 3a_1) = (1, 2, 3) = e_1+2e_2+3e_3$$

$$T_2(e_2) = (-a_2, 0, 2a_2) = (-1,0,2) = -e_1+0e_2+2e_3.$$

$$[T_2]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$$

Now,

$$[T_1]_\beta^\gamma + [T_2]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} \quad \text{— (1)}.$$

Now,

$(T_1+T_2)(a_1,a_2) = (2a_1+2a_2,\ 2a_1,\ 5a_1-2a_2)$.

$(T_1+T_2)(e_1) = (2a_1, 2a_1, 5a_1) = (2,2,5) = 2e_1+2e_2+5e_3$

$(T_1+T_2)(e_2) = (2a_2, 0, -2a_2) = (2,0,-2) = 2e_1+0e_2-2e_3$.

$$\therefore [T_1+T_2]_\beta^\gamma = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} \quad -(2)$$

$(1) = (2)$

$$\therefore [T_1+T_2]_\beta^\gamma = [T_1]_\beta^\gamma + [T_2]_\beta^\gamma$$

6) Find the linear transformation $T: V_3(R) \to V_3(R)$ determined by the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$ with respect to the standard basis $\{e_1, e_2, e_3\}$.

Soln

From the matrix

$T(e_1) = T(1,0,0) = (1,0,-1) = (a_1, 0, -a_1)$.

$T(e_2) = T(0,1,0) = (2,1,3) = (2a_2, a_2, 3a_2)$

$T(e_3) = T(0,0,1) = (1,1,4) = (a_3, a_3, 4a_3)$.

$\therefore T(a,b,c) = (a_1+2a_2+a_3,\ a_2+a_3,\ -a_1+3a_2+4a_3)$.

H.W

1) $T: V_2(R) \to V_3(R)$ gn by $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$ w.r to standard basis

2) $T: V_2(R) \to V_2(R)$ gn by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ w.r to S.B.

3) Construct the matrix $T: V_3(R) \to V(R)$ by $T(a_1,a_2,a_3) = 2a_1+a_2-3a_3$.

4) $T: V_3(R) \to V_3(R)$ gn by $T(a_1,a_2,a_3) = (2a_2+a_3,\ -a_1+4a_2+5a_3,\ a_1+a_3)$

5) Let $T: V_2(R) \to V_3(R)$ be def by
$T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$.
Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. compute $[T]_\beta^\gamma$.
If $\alpha = \{(1,2), (2,3)\}$ Compute $[T]_\alpha^\gamma$.

Ans $[T]_\beta^\gamma = \begin{bmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 1 \end{bmatrix}$ & $[T]_\alpha^\gamma = \begin{bmatrix} -7/3 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{bmatrix}$

7) Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

and $\beta = \{1, x, x^2\}$.

(i) If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$

(ii) If $T: P_2(R) \to R$ by $T(f(x)) = f(2)$ -
Compute $[T]_\beta^\alpha$

(iii) Define $T: P_2(R) \to M_{2\times2}(R)$ by $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$,
Compute $[T]_\beta^\alpha$.

Soln

i) Given $f(x) = 3 - 6x + x^2$ and $\beta = \{1, x, x^2\}$.

$f(x) = 3(1) - 6(x) + 1(x^2)$

$[f(x)]_\beta = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$

(ii) $T: P_2(R) \to R$ by $T(f(x)) = f(2)$ compute $[T]_\beta^\alpha$

$\beta = \{1, x, x^2\}$.

$T(1) = 1$

$T(x) = f(2) = 2$. ($\because f(x) = x$. rep x by 2)

$T(x^2) = f(2) = 4$ ($\because f(x) = x^2$)

$\therefore [T]_\beta^\alpha = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

(3) $T: P_2(R) \longrightarrow M_{2\times 2}(R)$.

$\beta = \{1, x, x^2\}$.

$T(1) \Rightarrow f(x) = 1$.

$$T(1) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

$T(x) \Rightarrow f(x) = x$

$$T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$T(x^2) \Rightarrow f(x) = x^2$

$$T(x) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

$$\therefore [T]_{\beta}^{\alpha} = \left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \right).$$

Sec 5

Eigen Values and Eigen Vectors

Left-Multiplication Transformation:-

Let A be an $m \times n$ matrix with entries from a field F. We denote by L_A, the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = A(x)$ where x is column vector $x \in F^n$.

$L_A(x) = Ax$ is called left multiplication transformation.

Diagonalizable:-

A linear operator T on a finite-dimensional V.S V is called diagonalizable, if there is an ordered basis β of V such that $[T]_\beta$ is a diagonal matrix.

A square matrix A is called diagonalizable, if L_A is diagonalizable.

Eigen Values and Eigen Vector of Linear transformation:-

Let T be a linear operator on a vector space V. A non zero vector $v \in V$ is called an eigen vector of T $\exists$ a scalar λ $\ni$ $T(v) = \lambda v$

The scalar λ is called the eigen value of A, corresponding to the eigen vector v.

Eigen Value and Eigen Vector of a Matrix A:-

Let A be in $M_{n\times n}(F)$. A non zero vector $v \in F^n$ is called an eigen vector of A if v is an eigen vector of L_A. (ie) $Av = \lambda v$ for some scalar λ.

The scalar λ is called the eigen value of A, corresponding to the eigen vector v.

Result:-

A linear operator T on a finite dimensional vector space V is a diagonalizable iff $\exists$ an ordered

basis β for V consisting of eigen vectors of T.
Further more, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigen vector of T and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigen value corresponding to v_j for $1 \leq j \leq n$.

Problems:-

1) Let T be the linear operator on $P_2(R)$ defined by $T[f(x)] = f(x) + (x+1)f'(x)$. Let β be the standard ordered basis for $P_2(R)$ and determine whether β is a basis consisting of eigen vectors of T. Also check T is diagonalizable.

Soln:-

Here $\beta = \{1, x, x^2\}$ be the standard basis

then,

$$T(1) = 1 + (x+1)(0) = 1 = 1(1) + 0(x) + 0(x^2)$$

$$T(x) = x + (x+1)(1) = 2x+1 = 1(1) + 2(x) + 0(x^2)$$

$$T(x^2) = x^2 + (x+1)(2x) = 3x^2 + 2x = 0(1) + 2(x) + 3(x^2)$$

$\therefore$ The corresponding matrix is

$$[T]_\beta = A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic eqn is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 6; \quad S_2 = \begin{matrix} 6 \\ +3+2 \\ = 11 \end{matrix}; \quad S_3 = 6.$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3.$$

To find eigen Vectors,

$$[A-\lambda I]X = 0.$$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(1).$$

Case 1

$\lambda = 1$

✓ $0x_1 + x_2 + 0x_3 = 0$

✓ $0x_1 + x_2 + 2x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0.$

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0} \div \frac{1}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Case 2

$\lambda = 2$

✓ $-x_1 + x_2 + 0x_3 = 0$

✓ $0x_1 + 0x_2 + 2x_3 = 0$

$0x_1 + 0x_2 + x_3 = 0.$

$$\begin{matrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & 0 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0} \div \frac{1}{2}$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case 3

$\lambda = 3$

✓ $-2x_1 + x_2 + 0x_3 = 0$

✓ $0x_1 - x_2 + 2x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$$\begin{matrix} 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2} \div \frac{1}{2} \qquad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Now $\gamma = \{1, 1+x, 1+2x+x^2\}$ is a ordered basis for $P_2(R)$

Thus T is diagonalizable

and $[T]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2) Find the eigen Values of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and eigen Vectors.

Soln

$\lambda^2 - S_1\lambda + S_2 = 0.$

$S_1 = 1+1 = 2$

$S_2 = 1$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1, 1.$

$\therefore$ The eigenValues are $1, 1$.

To find eigen Vectors,

$[A - \lambda I]X = 0.$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad -(1).$$

for $\lambda = 1$,

$0x_1 + 2x_2 = 0.$

$0x_1 + 0x_2 = 0.$

the system does not contains eigen Vectors.

3) Find the eigen Value and eigen Vectors of the matrix. $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Soln

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_1 = 2$

$S_2 = 1+1 = 2.$

$\lambda^2 - 2\lambda + 2 = 0.$

$$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i.$$

$\lambda = 1+i,\ 1-i$ are eigen Values.

To find eigen Vector,

$$[A - \lambda I]X = 0.$$

$$\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{--- } (1)$$

Case 1

$\lambda = 1+i$

Sub in (1)

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 i + x_2 = 0.$

$\cancel{+}x_1 i = \cancel{+}x_2$

$$\frac{x_1}{1} = \frac{x_2}{i}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Case 2

$\lambda = 1 - i.$

Sub in (1).

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i x_1 + x_2 = 0.$$

$$i x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{i}$$

$$X_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}.$$

H.W

1) Find the eigen Values and eigen Vectors of following matrices

(i) $\begin{bmatrix} 6 & 10 \\ 14 & 25 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\lambda = 2, 3, 5.$

EV : $\lambda = \dfrac{31 \pm \sqrt{921}}{2}$ $\lambda = 5, 1$

(iv) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ (V) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ (Vi) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

$\lambda = 1, 5$ $\lambda = 1, 3, -4$

$\begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$

4) for each of the following linear operator T on a V.S V and ordered bases β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigen Vector of T.

ⓐ $V = V_2(\mathbb{R})$; $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$

Soln

Let $T(a, b) = (10a - 6b, 17a - 10b)$

$T(1, 2) = (10 - 12, 17 - 20) = (-2, -3$

$T(2,3) = (20-18,\ 34-30)$

$= (2,4)$

Now we express $(-2,-3)$ as a linear combination of $(1,2)$ and $(2,3)$.

$\Rightarrow (-2,-3) = a_1(1,2) + a_2(2,3)$

$a_1 + 2a_2 = -2$.— ①

$2a_1 + 3a_2 = -3$.— ② .

$$
\begin{aligned}
①\times 2 \Rightarrow\ & \cancel{2a_1} + 4a_2 = -4 \\
& \cancel{2a_1} + 3a_2 = -3 \\
& (-)\quad (-)\quad (+) \\
\hline
& a_2 = -1
\end{aligned}
$$

Sub in ①

$a_1 - \cancel{2} = -\cancel{2}$

$a_1 = 0$.

Illarly,

$(2,4) = a_1(1,2) + a_2(2,3)$

$a_1 + 2a_2 = 2$ — ③

$2a_1 + 3a_2 = 4$ — ④

$$
\begin{aligned}
③\times 2 \Rightarrow\ & \cancel{2a_1} + 4a_2 = 4 \\
& \cancel{2a_1} + 3a_2 = 4 \\
& (-)\quad (-)\quad (-) \\
\hline
& a_2 = 0 .
\end{aligned}
$$

Sub in ③

$\boxed{a_1 = 2}$

$\therefore [T]_\beta = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} = A.$

The characteristic polynomial is

$\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 0,\ S_2 = 0-(-2) = 2$

$\lambda^2 + 2 = 0$

$\lambda^2 = -2$

$\lambda = \pm i\sqrt{2}.$

$\lambda = i\sqrt{2}, -i\sqrt{2}.$

To find eigen vector,

$[A - \lambda I]X = 0.$

$$\Rightarrow \begin{bmatrix} -\lambda & 2 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1

$\lambda = i\sqrt{2}$

$-i\sqrt{2}x_1 + 2x_2 = 0.$

$-x_1 - i\sqrt{2}x_1 = 0.$

$\cancel{+} i\sqrt{2}x_1 = \cancel{+} 2x_2$

$$\frac{x_1}{\frac{2}{\sqrt{2}}} = \frac{x_2}{i\sqrt{2}}$$

$$X_1 = \begin{bmatrix} \sqrt{2} \\ i \end{bmatrix}$$

Case 2

$\lambda = -i\sqrt{2}.$

$i\sqrt{2}x_1 + 2x_2 = 0.$

$i\sqrt{2}x_1 = -2x_2$

$$\frac{x_1}{\frac{-2}{\sqrt{2}}} = \frac{x_2}{i\sqrt{2}}.$$

$$X_2 = \begin{bmatrix} -\sqrt{2} \\ i \end{bmatrix}$$

H.W

1) $V = P_1(R)$; $T(a+bx) = (6a-6b) + (12a-11b)x$ and

$B = \{3+4x, 2+3x\}$

$\lambda = -2, -3$

Ans $A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Sec 6

Diagonalizability :-

Result :-

1) Let T be a linear operator on a V.S. V and $\lambda_1, \lambda_2, \ldots, \lambda_K$ be distinct eigen Values of T. If $v_1, v_2, \ldots, v_K$ are Eigen Vector of $T \ni \lambda_i$ Corresponds to v_i, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

2) Let T be a linear operator on an n-dimensional V.S. 'V.' If T has 'n' distinct eigen Values, then T is diagonolizable. But the converse is not true.

Defn

A Polynomial $f(t)$ in $P(F)$ is said to splits over F if there are Scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in $F \ni$.

$$f(t) = c(t-a_1)(t-a_2) \ldots (t-a_n).$$

Algebric Multiplicity

Let λ be an eigen Value of a linear operator or a matrix, with characteristic polynomial $f(t)$.

The algebraic multiplicity of λ is the largest +ve integer K, for which $(t-\lambda)^k$ is a factor of $f(t)$

Problems

1) Consider the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

The characteristic polynomial is

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0.$$

The characteristic polynomial is

$$= -(\lambda-1)(\lambda-2)(\lambda-3).$$

$\lambda = 1,2,3$ is an eigen Values of A with multiplicity 1.

H.W.

1) $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$ Ans $\lambda = 3$ is of multiplicity 2
$\lambda = 4$ is of multiplicity 1

Eigen Space:-

Let T be a linear operator on a Vector Space V and let λ be an eigen Value of T.

then, $E_\lambda = \{ x \in V / T(x) = \lambda x \} = N(T - \lambda I)$

Procedure for diagnalizability:-

1) choose the basis β for a Vector Space V
2) obtain the matrix $A = [T]_\beta$
3) characteristic polynomial of A
4) (i) The characteristic polynomial Splits
   (ii) for each repeated eigen Value (λ) verify the condition

$$\text{Multiplicity of } \lambda = n - \text{rank}(A - \lambda I).$$

where $n \to$ number of unknowns.

5) If the above condition is Satisfied, then the matrix A is diagonalizable and hence the linear transformation T is diagonalizable.

Note

Let A be an $n \times n$ diagonilizable matrix. then we can find an invertible matrix. Q

$\ni D = Q^{-1} A Q$.

Problems :-

1) Let the linear operator $T : P_2(R) \to P_2(R)$ defined by $T(f(x)) = f'(x)$. With respect to standard ordered basis β of $P_2(R)$ check T is diagonalizable or not? Also, find its eigen space and its dimension.

Soln

$\beta = \{1, x, x^2\}$ be the standard basis.

then, $T(1) = 0 = 0(1) + 0(x) + 0(x^2)$

$T(x) = 1 = 1(1) + 0(x) + 0(x^2)$

$T(x^2) = 2x = 0(1) + 2(x) + 0(x^2)$

then the corresponding matrix

$$A = [T]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

To find eigen values,

The characteristic eqn is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

$$S_1 = 0, \quad S_2 = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 0.$$

$$S_3 = 0(0) - 1(0) + 0(0) = 0.$$

$$\Rightarrow \lambda^3 = 0.$$

$$\Rightarrow \lambda = 0, 0, 0.$$

$\therefore \lambda = 0$ is eigen value of multiplicity 3.

$n = 3$ (no. of unknowns).

$$(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix}$$

When $\lambda = 0$.

$$[A-\lambda I] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore |A-\lambda I| = 0$ rank $(A-\lambda I) \neq 3$.

the minor $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$.

rank $(A-\lambda I) = 2$.

Now,

$n -$ rank $(A-\lambda I)$ = multiplicity of λ.

$3 - 2 = 3$.

$1 \neq 3$.

$\therefore$ T is not diagonalizable.

To find Eigen spaces,

$[A-\lambda I][X] = 0$.

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(1).$$

When $\lambda = 0$

$0x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 2x_3 = 0$.

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0} \div \frac{1}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Eigen Space $E\lambda_1 = \{1 + 0.x + 0x^2\}$ is a constant polynomial.

So, $\{1\}$ is a basis of $E\lambda$.

dim $E\lambda = 1$.

Also, $\dim(E\lambda) \neq$ multiplicity of λ

$\therefore$ T is not diagonalizable.

2). Let T be the linear operator on $V_3(R)$ defined by $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1+3a_2+2a_3 \\ a_1+4a_3 \end{pmatrix}$ and β be the Standard basis for $V_3(R)$. Determine the eigen space of T, corresponds to each eigen Value.

Soln

Given,

$T(a_1, a_2, a_3) = (4a_1+a_3, 2a_1+3a_2+2a_3, a_1+4a_3)$

Let $\beta = (e_1, e_2, e_3)$ be the Standard basis of $V_3(R)$.

then, $T(e_1) = (4a_1, 2a_1, a_1) = (4,2,1) = 4e_1+2e_2+e_3$

$T(e_2) = (0a_2, 3a_2, 0a_2) = (0,3,0) = 0e_1+3e_2+0e_3$

$T(e_3) = (a_3, 2a_3, 4a_3) = (1,2,4) = e_1+2e_2+4e_3$

$\therefore$ the matrix of β in $[T]_\beta = A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

To find eigen Values,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 11,$

$S_2 = \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 2 & 3 \end{vmatrix}$

$= 12 + 15 + 12.$

$= 39.$

$S_3 = 4(12) - 0 + 1(-3)$

$= 48 - 3 = 45.$

$\Rightarrow \lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0.$

$\lambda = 5, 3, 3.$

$\lambda = 5$ is a eigen Value of multiplicity 1
$\lambda = 3$ is a eigen Value of multiplicity 2.

$\because$ for $\lambda = 5$, the multiplicity is 1.
the condition for diagnolization is always true.

$\therefore$ It is enough to check for 3.

multiplicity of $\lambda = 3 = 2$.
$n = 3$ (no. of unknowns)

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$|A - \lambda I| = 1(0) + 0 - 1(0) = 0$ $\therefore$ Rank $(A-\lambda I) \neq 3$.
minor of $A - \lambda I = 0$. $\therefore$ Rank $(A - \lambda I) \neq 2$.

$\therefore$ Rank $(A - \lambda I) = 1$.

$2 = 3 - 1 = 2$.

$\therefore$ Therefore T is diagnolizable.

To find Eigen Space,

$[A - \lambda I] X = 0$.

$$\begin{bmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ 1 & 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

When $\lambda = 5$ in (1)

$-x_1 + 0x_2 + x_3 = 0$
$2x_1 - 2x_2 + 2x_3 = 0$
$x_1 + 0x_2 - x_3 = 0$

$$\begin{matrix} 0 & 1 & -1 & 0 \\ -2 & 2 & 2 & -2 \end{matrix}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2} \div \frac{1}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

~~for~~ $\therefore E\lambda_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. dim $E\lambda_1 = 1$.

for $\lambda = 3$,

(1) becomes

$$\begin{aligned} x_1 + 0x_2 + x_3 &= 0 \\ 2x_1 + 0x_2 + 2x_3 &= 0 \\ x_1 + 0x_2 + x_3 &= 0 \end{aligned}$$

$$x_1 + 0x_2 + x_3 = 0.$$

Put $x_1 = 0 \Rightarrow 0x_2 = -x_3$.

$$\frac{x_2}{-1} = \frac{x_3}{0}.$$

$$E\lambda_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Put $x_2 = 0 \Rightarrow x_1 + x_3 = 0$

$$x_1 = -x_3$$

$$\frac{x_1}{-1} = \frac{x_3}{1}$$

$$E\lambda_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen Space $E\lambda_2 = \left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow$ dim $E\lambda_2 = 2$.

3) Test the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$ for diagonalizability. If A is diagonalizable, find an invertible matrix Q and a diagonal matrix D $\ni$ $Q^{-1} A Q = D$. Also, using this result, compute A^n for any +ve integer n.

Soln

Given $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$

The characteristic eqn is

$$\lambda^2 - S_1 \lambda + S_2 = 0$$

$$S_1 = 0 + 3 = 3$$

$$S_2 = 0 + 2.$$

$$\lambda^2 - 3\lambda + 2 = 0.$$

$$\lambda = 1, 2.$$

2

-3

-2 -1

$\lambda = 1$ is the eigen Value with multiplicity 1

$\lambda = 2$ is the eigen Value with multiplicity 1

To find eigen space,

$$[A - \lambda I] X = 0.$$

$$\begin{bmatrix} 0-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When $\lambda = 1$

$$-x_1 - 2x_2 = 0.$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1}$$

$$E\lambda_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for $\lambda = 2$,

$$-2x_1 - 2x_2 = 0.$$

$$x_1 + x_2 = 0.$$

$$x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$E\lambda_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$\therefore \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a ordered basis for R^2.

Let $Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

then $Q^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

Now, $D = Q^{-1} A Q$.

$$AQ = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}.$$

$$Q^{-1} A Q = \frac{1}{-1} \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} -2+1 & -2+2 \\ 2-2 & 2-4 \end{pmatrix}$$

$$= \frac{1}{-1} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Now,

$$A^n = (QDQ^{-1})^n$$

$$= (QDQ^{-1})(QDQ^{-1}) \dots (QDQ^{-1})$$

$$= QD^nQ^{-1}$$

$$= Q\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} Q^{-1}$$

$$QD^n = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} = \begin{pmatrix} -2 & -2^n \\ 1 & 2^n \end{pmatrix}$$

$$QD^nQ^{-1} = \begin{pmatrix} -2 & -2^n \\ 1 & 2^n \end{pmatrix}\begin{pmatrix} -1 & -1 \\ +1 & +2 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2^n & 2-2^{n+1} \\ -1+2^n & -1+2^{n+1} \end{pmatrix}$$

H.W

1) Test the matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ for diagonalizability.

Ans $\lambda = 4,3,3$ (Not diagnolizable).

2) Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$ and $\beta = \{1, x, x^2\}$ be the standard basis. Test the diagonalizability of T.

Ans $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ $\lambda = 2,1,1$ (diagnolizable).