UNIT-1

VECTOR SPACES

Sec 1

Vector Space (Linear Space):

A Vector Space V over a field F consists of a set on Which two operations (called addition & Scalar multiplication respectively) are defined So that for each pair $u, v \in V$, $\exists$ unique element $u+v \in V$

& for each element $\alpha \in F$ and $u \in V$

$\exists$ unique element $\alpha u \in V$,

$\ni$, the following Conditions hold (Axioms)

(i) Commutative property:

$\forall\ u, v \in V$,

$$u+v = v+u$$

(ii) Associative property:

$\forall\ u, v, w \in V$,

$$(u+v)+w = u+(v+w)$$

(iii) $\exists$ element $0 \in V$

$\ni\ u+0 = u \quad \forall\ u \in V$

(iv) For each element, $u \in V$

$\exists$ an element, $v \in V$

$\ni\ u+v = 0$ (Inverse element)

(v) For each element, $u \in V$

$$1 \cdot u = u$$

(vi) for each Pair of elements $\alpha, \beta \in F$ and each element $u \in V$

$$(\alpha\beta)u = \alpha(\beta u)$$

(vii) For each element $\alpha \in F$ & each pair of elements $u, v \in V$,

$$\alpha(u+v) = \alpha u + \alpha v$$

(viii) For each pair of elements $\alpha, \beta \in F$ and each element $u \in V$,

$$(\alpha+\beta)u = \alpha u + \beta u$$

Note

1) The elements of field F is called Scalars
2) The elements of Vector Space V is called Vectors
3) for each $\alpha \in F$ & $v \in V$
   αv is called Scalar multiplication.

$\therefore$ Scalar multiplication gives rise to a function

$$F \times V \to V$$

def by $(\alpha, v) \to \alpha v$.

Theorem 1 (Cancellation law for Vector addition)

If x, y and z are Vectors in Vector Space $V \ni x+z = y+z$ then $x = y$

Proof:

Given $x+z = y+z$ —① .

$\because$ V is a Vector Space,

by Inverse axiom,

$\forall z \in V$, $\exists$ a vector $v \in V$

$\ni z + v = 0$ —②.

Now, consider $x = x + 0$

$= x + (z+v)$ (by ②)

$= (x+z)+v$ (by Axiom ii)

$= (y+z)+v$ (by 1)

$= y+(z+v)$ (by Axiom ii)

$= y+0$ (by 2)

$= y$

$\Rightarrow \boxed{x=y}$

Theorem 2

In any vector Space V, the following Statements are true.

(a) $0.x = 0 \quad \forall\, x \in V$

(b) $(-\alpha)x = -(\alpha x) = \alpha(-x)$ for each $\alpha \in F$ and each $x \in V$

(c) $\alpha.0 = 0$ for each $\alpha \in F$.

Proof

(a) Consider

$0x + 0x = (0+0)x$ (by viii).

$= 0x$

$= 0x + 0$ (by iii).

$= 0 + 0x$ (by i).

$\Rightarrow 0x + 0x = 0 + 0x$

$\Rightarrow 0x = 0$ (by ~~left~~ cancellation law)

(b) by (a)

$0 = 0x$

$= [\alpha + (-\alpha)]x$ (by (iv) $-\alpha$ is the inverse element of $\alpha \in F$)

$$0 = \alpha x + (-\alpha) x \quad \text{(by viii)}$$

$$\Rightarrow (-\alpha) x = -(\alpha x) \quad \text{①}$$

Also,

$$0 = \alpha 0$$

$$= \alpha[x + (-x)] \quad \text{(by (iv) } -x \text{ is the inverse of } x \in V\text{)}$$

$$= \alpha x + \alpha(-x) \quad \text{(by viii)}$$

$$0 = \alpha x + \alpha(-x)$$

$$\alpha(-x) = -(\alpha x) \quad \text{②}$$

from ① & ②

$$-(\alpha x) = (-\alpha) x = \alpha(-x)$$

(c) Consider

$$\alpha 0 + \alpha 0 = \alpha(0+0) \quad \text{(by vii)}$$

$$= \alpha 0$$

$$= \alpha 0 + 0 \quad \text{(by iii)}$$

$$\Rightarrow \alpha 0 + \alpha 0 = \alpha 0 + 0$$

$$\alpha 0 = 0 \quad \text{(by cancellation law)}.$$

Problems

1) $\mathbb{R} \times \mathbb{R}$ is a Vector Space over $\mathbb{R}$ under usual addition and Scalar multiplication defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$\& \quad \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

Soln

(i) Commutative property:-

$$(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$= (y_1 + x_1, y_2 + x_2) \text{ (by (i))}$$
$$= (y_1, y_2) + (x_1, x_2)$$

$\therefore$ $\mathbb{R} \times \mathbb{R}$ is commutative.

(ii) Associative property:-

$(x_1, x_2), (y_1, y_2), (z_1, z_2) \in (\mathbb{R} \times \mathbb{R})$.

Consider,

$$((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)$$
$$= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$$
$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$$
$$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2)) \text{ (by (ii))}$$
$$= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$$
$$= (x_1, x_2) + ((y_1, y_2) + (z_1, z_2))$$

$\therefore$ $\mathbb{R} \times \mathbb{R}$ is Associative.

(iii) $\exists$ element $(0, 0) \in \mathbb{R} \times \mathbb{R}$

$\ni (x_1, x_2) + (0, 0)$
$= (x_1 + 0, x_2 + 0)$
$= (x_1, x_2)$ (by iii)
$\in \mathbb{R} \times \mathbb{R}$ $\quad \forall (x_1, x_2) \in \mathbb{R} \times \mathbb{R}$.

(iv) For each element $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$

$\exists$ an element $(-x_1, -x_2) \in \mathbb{R} \times \mathbb{R}$

$$\exists\ (x_1, x_2) + (-x_1, -x_2)$$
$$= (x_1 - x_1,\ x_2 - x_2)$$
$$= (0, 0) \text{ (by iv)}$$
$$\in \mathbb{R} \times \mathbb{R}.$$

(V) For each element, $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$

$$\exists\ 1 \cdot (x_1, x_2) = (1 \cdot x_1,\ 1 \cdot x_2) \text{ (by V)}.$$
$$= (x_1, x_2) \in \mathbb{R} \times \mathbb{R}.$$

(Vi) for each Pair of elements $\alpha, \beta \in F$
and $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$

$$\alpha\beta\ (x_1, x_2) = (\alpha\beta(x_1), \alpha\beta(x_2)) \text{ (by given Condition)}$$
$$= (\alpha(\beta x_1), \alpha(\beta x_2)) \text{ (by Vi)}.$$
$$= \alpha\ ((\beta x_1), (\beta x_2)) \text{ (by given Condition)}$$
$$= \alpha(\beta(x_1, x_2))$$

(Vii) for each element, $\alpha \in F$.
and $(x_1, x_2), (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$.

$$\alpha\ [(x_1, x_2) + (y_1, y_2)]$$
$$= \alpha\ (x_1 + y_1,\ x_2 + y_2) \text{ (by gn)}.$$
$$= (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) \text{ (by gn)}$$
$$= (\alpha x_1 + \alpha y_1,\ \alpha x_2 + \alpha y_2)\ ($$
$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2).$$
$$= \alpha(x_1, x_2) + \alpha(y_1, y_2).$$

(viii) For each pair of elements $\alpha, \beta \in F$ and each element $(x_1, x_2) \in V$

$(\alpha+\beta)(x_1, x_2)$

$= ((\alpha+\beta)x_1, (\alpha+\beta)x_2)$ (by given condition)

$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2)$ (by viii)

$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$

$= \alpha(x_1, x_2) + \beta(x_1, x_2)$

$\therefore$ all the axioms of Vector Space are Satisfied, $R \times R$ is a Vector Space over R,

2) Consider $V = \{p(x) = a_0x^2 + a_1x + a_2 / a_0, a_1, a_2$ are real numbers$\}$ the set of all polynomials of degree ≤ 2. Then, V is a V.S over F under usual addition and multiplication.

Soln Let $p(x), q(x), r(x) \in V$ and $\alpha \in F$

Where $p(x) = a_0x^2 + a_1x + a_2$

$q(x) = b_0x^2 + b_1x + b_2$

$r(x) = c_0x^2 + c_1x + c_2$

(i) Commutative Property:-

$(p+q)(x) = p(x) + q(x)$

$= a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2$

$= (a_0+b_0)x^2 + (a_1+b_1)x + a_2 + b_2$

$= (b_0+a_0)x^2 + (b_1+a_1)x + b_2 + a_2$

$= b_0x^2 + b_1x + b_2 + a_0x^2 + a_1x + a_2$

$= q(x) + p(x)$

$= (q+p)(x)$.

(ii) Associative Property:-

$(p+q)(x)+r(x)$

$= [(a_0+b_0)x^2+(a_1+b_1)x+(a_2+b_2)] + [c_0x^2+c_1x+c_2]$

$= ((a_0+b_0)+c_0)x^2+((a_1+b_1)+c_1)x+((a_2+b_2)+c_2)$

$= (a_0+(b_0+c_0))x^2+(a_1+(b_1+c_1))x+(a_2+(b_2+c_2))$

$= (a_0x^2+a_1x+a_2)+((b_0+c_0)x^2+(b_1+c_1)x+b_2+c_2)$

$= p(x)+(q+r)(x)$.

(iii) $\exists$ polynomial ~~element~~ $0 = 0x^2+0x+0 \in V$

$\ni p(x)+0 = a_0x^2+a_1x+a_2+0x^2+0x+0$

$= (a_0+0)x^2+(a_1+0)x+a_2+0$

$= a_0x^2+a_1x+a_2$

$= p(x) \quad \forall\ p(x) \in V$

(iv) for each polymial, $p(x) \in V$

$\exists$ a polynomial, $-p(x) \in V$

$\ni p(x)-p(x) = a_0x^2+a_1x+a_2-a_0x^2-a_1x-a_2$

$= (a_0-a_0)x^2+(a_1-a_1)x+a_2-a_2$

$= 0x^2+0x+0 = 0$

(v) For each polynomial, $p(x) \in V$

$1\cdot p(x) = 1\,(a_0x^2+a_1x+a_2)$

$= (1\cdot a_0)x^2+(1\cdot a_1)x+1\cdot a_2$

$= a_0x^2+a_1x+a_2$

$= p(x)$

(vi) for each pair of elements,

$\alpha, \beta \in F$

and each polynomial $p(x) \in V$

$$(\alpha\beta) p(x) = (\alpha\beta)(a_0x^2 + a_1x + a_2)$$

$$= (\alpha\beta) a_0x^2 + (\alpha\beta) a_1x + (\alpha\beta) a_2$$

$$= \alpha(\beta a_0)x^2 + \alpha(\beta a_1)x + \alpha(\beta a_2)$$

$$= \alpha(\beta a_0x^2 + \beta a_1 x + \beta a_2)$$

$$= \alpha(\beta p(x))$$

(vii) For each element $\alpha \in F$ & each pair of ~~ab~~ polymials $p(x), q(x) \in V$

$$\alpha(p(x) + q(x)) = \alpha(a_0x^2 + a_1x + a_2 + b_0x^2 + b_1x + b_2)$$

$$= \alpha[(a_0 + b_0)x^2 + (a_1 + b_1)x + a_2 + b_2]$$

$$= [(\alpha a_0 + \alpha b_0)x^2 + (\alpha a_1 + \alpha b_1)x + \alpha a_2 + \alpha b_2]$$

$$= \alpha a_0x^2 + \alpha a_1x + \alpha a_2 + \alpha b_0x^2 + \alpha b_1x + \alpha b_2$$

$$= \alpha p(x) + \alpha q(x)$$

(viii) For each pair of elements $\alpha, \beta \in F$ and each ~~ab~~ polynomial $p(x) \in V$

$$(\alpha + \beta) p(x) = (\alpha + \beta)(a_0x^2 + a_1x + a_2)$$

$$= (\alpha + \beta) a_0x^2 + (\alpha + \beta) a_1x + (\alpha + \beta) a_2$$

$$= \alpha a_0x^2 + \alpha a_1x + \alpha a_2 + \beta a_0x^2 + \beta a_1x + \beta a_2$$

$$= \alpha p(x) + \beta p(x).$$

$\therefore$ Set of all polynomial of degree ≤ 2 is a Vector Space.

Note

In general, $P_n(F)$ is a Vector Space.

3) The set of all $m \times n$ matrices with entries from a field F is a vector space, which we denote by $M_{m\times n}(F)$, with the following operations of matrix addition and scalar multiplication.

Soln

for $A, B \in M_{m\times n}(F)$ & $c \in F$

then, $(A+B)_{ij} = A_{ij} + B_{ij}$

$(cA)_{ij} = cA_{ij}$

for $1 \leq i \leq m$ & $1 \leq j \leq n$.

(i) $\forall$ $A, B \in M_{m\times n}$

$$A+B = (A+B)_{ij} = A_{ij} + B_{ij}$$
$$= B_{ij} + A_{ij}$$
$$= (B+A)_{ij} = B+A$$

(ii) $\forall$ $A, B, C \in M_{m\times n}$

$$A + (B+C) = (A+(B+C))_{ij} = A_{ij} + (B+C)_{ij}$$
$$= A_{ij} + (B_{ij} + C_{ij})$$
$$= (A_{ij} + B_{ij}) + C_{ij}$$
$$= (A+B)_{ij} + C_{ij}$$
$$= ((A+B)+C)_{ij}$$
$$= (A+B)+C.$$

(iii) $\exists$ matrix $O \in M_{m\times n}$

$$\ni A+O = (A+O)_{ij}$$
$$= A_{ij} + O_{ij} = A_{ij}$$
$$= A.$$

(iv) For each matrix, $A \in M_{m\times n}$

$\exists$ an matrix $-A \in M_{m\times n}$

$$\ni A-A = (A-A)_{ij}$$
$$= A_{ij} - A_{ij} = O_{ij}$$
$$= O$$

(v) For each ~~elements~~ matrix, $A \in M_{m\times n}$

$$1.A = 1.A_{ij} = A_{ij} = A.$$

(vi) For each pair of elements $\alpha, \beta \in$ ~~$M_{m\times n}$~~ F and each element of matrix $A \in M_{m\times n}$

$$(\alpha\beta)A = (\alpha\beta)A_{ij}$$
$$= (\alpha\beta A)_{ij}$$
$$= \alpha(\beta A)_{ij}$$
$$= \alpha(\beta A_{ij}) = \alpha(\beta A)$$

(vii) For each element $\alpha \in F$ & each pair of matrix $A, B \in M_{m\times n}$

$$\alpha(A+B) = \alpha(A+B)_{ij}$$
$$= \alpha(A_{ij} + B_{ij})$$
$$= \alpha A_{ij} + \alpha B_{ij}$$
$$= (\alpha A)_{ij} + (\alpha B)_{ij}$$
$$= \alpha A + \alpha B.$$

(viii) For each pair of elements $\alpha, \beta \in F$ and each matrix $A \in M_{m\times n}$.

$$(\alpha+\beta) A = (\alpha+\beta) A_{ij}$$
$$= \alpha A_{ij} + \beta A_{ij}$$
$$= \alpha A + \beta A.$$

$\therefore M_{(m\times n)} F$ is a Vector Space over the field F.

<u>H.W</u>

1) Let $V = \{a + b\sqrt{2} \,/\, a, b \in Q\}$. Then V is a Vector Space over Q under usual addition and multiplication.

4) $\mathbb{R}$ is not a Vector Space over C.

<u>Soln</u>

Here, addition '+' Satisfies all the Conditions, But the Scalar multiplication $\alpha \in C$
$\Rightarrow \alpha = a + ib$

$$\alpha u = (a+ib) u = au + ibu \notin \mathbb{R} \quad \text{Where } u \in \mathbb{R}$$

$\therefore$ R is not a Vector Space over C.

5) Let V be the Set of all ordered pairs of real numbers. Addition and multiplication are defined by $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ and $\alpha(x, y) = (x, \alpha y)$ Where $x, x_1, y, y_1, \alpha \in \mathbb{R}$. Then V is not a vector Space over $\mathbb{R}$.

<u>Soln</u>

Here, addition '+' Satisfies all the Conditions.

But, $(\alpha+\beta) u \neq \alpha u + \beta u$ Wher $u = (x, y)$

Consider, $(\alpha+\beta) u = (\alpha+\beta)(x, y)$.
$$= (x, (\alpha+\beta) y)$$

$$= (x, \alpha y + \beta y) \quad \text{(1)}$$

Also,

$$\alpha u + \beta u = \alpha (x, y) + \beta (x, y)$$
$$= (x, \alpha y) + (x, \beta y)$$
$$= (x + x, \alpha y + \beta y)$$
$$= (2x, \alpha y + \beta y) \quad \text{(2)}$$

$$(1) \neq (2)$$

$\therefore$ $(\alpha + \beta) u \neq \alpha u + \beta u$

$\therefore$ V is not a vector space.

Sec 2 :-
SubSpaces :-

Defn

Let V be a Vector Space Over a field F. A non-empty Subset W of V is called Subspace of V, if W itself is a Vector Space Over F under the Operation of V

Note

1) for any V.S V,
V and $\{0\}$ is a Subspace of V. and $\{0\}$ is called trivial Subspace of V and
It is also called zero Subspace of V.

2) T.P W is a Subspace of V.
It is enough to prove

(i) $x+y \in W$ $(x, y \in W)$ (W is closed under addition.)

(ii) $cx \in W$ $(c \in F, x \in W)$ (W is closed under Scalar multiplication)

(iii) W has a zero Vector

(iv) Each Vector in W has an additive inverse in W.

Theorem 1 :-

Let V be a V.S and W a Subs~~pace~~et of V. Then W is a Subspace of V iff the following 3 conditions hold for the operation defined in V.

(a) $0 \in W$ (b) $x+y \in W$ $(x, y \in W)$ (c) $cx \in W$ $(x \in W, c \in F)$

Proof

Let W be a Subspace of V.

Then, W itself is a Vector Space with the operations of addition and Scalar multiplication def on V, and hence W is closed w.r.to Vector addition and Scalar multiplication.

$\therefore$ (b) and (c) is true.

$\therefore$ W is a Subspace, $\exists$ a vector $0' \in W$
$\Rightarrow x + 0' = x \ \forall x \in W$

But also, $x+0=x$

$\therefore x = x+0 = x+0'$

$\Rightarrow x+0 = x+0'$

$\Rightarrow 0 = 0'$ (cancellation law).

$\Rightarrow 0 \in W$

$\therefore$ (a) is also Satisfied.

Conversely,

Let (a), (b), (c) are hold.

We have to prove,

W is a Subspace of V.

from (a), (b), (c),

W is closed under addition, Scalar multiplication and $0 \in W$. So it is enough to prove the inverse exist in W.

$\therefore$ W is non-empty.

$\exists$ an element $x \in W$

$\Rightarrow (-1)x \in W$ (by ©).

$\Rightarrow -x \in W$.

Thus, W contains additive inverse

$\Rightarrow$ W is a Subspace of V.

Theorem 2

Let V be a Vector Space over a field F. A non-empty Subset W of V is a Subspace of V iff $x, y \in W$ and $\alpha, \beta \in F$ then, $\alpha x + \beta y \in W$

Proof:-

Let W be a Subspace of V and let $x, y \in W$ and $\alpha, \beta \in F$

then, by Previous thm 1, (a), (b), (c) holds

By ©, $\alpha x \in W$ and $\beta y \in W$

By (b), $\alpha x+\beta y \in W$.

Conversely, if $x, y \in W$ and $\alpha, \beta \in F$
then, $\alpha x+\beta y \in W$.

We have to prove,
W is a Subspace of V.

taking $\alpha=\beta=1$, then $x+y \in W$

taking $\beta=0$, $\alpha \in F$
then, $\alpha x \in W$.

taking $\alpha=\beta=0$, then $0 \in W$.

taking $\alpha=-1$, $\beta=0$ then $-x \in W$.

$\therefore$ W is a Subspace of V.

Examples for Subspace:-

1) $W=\{(a,0,0) / a \in R\}$ is a Subspae of $V_3(R)$

Soln

Let $x=(a,0,0)$; $y=(b,0,0) \in W$
and $\alpha, \beta \in R$

then $\alpha x+\beta y = (\alpha a,0,0)+(\beta b,0,0)$
$= (\alpha a+\beta b,0,0) \in W$.

$\therefore$ W is a Subspace of $V_3(R)$

2) $W=\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} / a, b \in R\right\}$ is a Subspace of $M_{2\times 2}(R)$

3) $P_n(F)$, polynomial of degree $\leq n$ is a Subspace of $P(F)$

4) The Set of diagonal matrices $(n\times n)$ is a subspace of $M_{n\times n}(F)$

(5) The Set of $n \times n$ matrice having trace equal to zero is a Subspace of $M_{n\times n}(F)$.

(6) The set of $m \times n$ matrices having non-negative entries is not a Subspace of $M_{m\times n}(R)$.

Theorem 3 :-

P.T, the intersection of two Subspace of a Vector Space is a Subspace.

Proof

Let A and B be any 2 Subspaces of a V.S V over a field F.

To. Prove, $A \cap B$ is also Subspace of V.

Clearly $0 \in A \cap B$ and hence $A \cap B$ is non-empty.

Let $x, y \in A \cap B$ and $\alpha, \beta \in F$

then, $x, y \in A$ and $x, y \in B$.

$\because$ A and B are Subspaces,

$\alpha x + \beta y \in A$ and $\alpha x + \beta y \in B$

$\Rightarrow \alpha x + \beta y \in A \cap B$.

$\therefore A \cap B$ is a Subspace of V.

Theorem 4 :-

P.T, the Union of two Subspace of a Vector Space need not be a Subspace.

Proof

The above theorem can be proved by an example.

Let $A = \{ (a, 0, 0) / a \in \mathbb{R} \}$

and $B = \{ (0, b, 0) / b \in \mathbb{R} \}$

Clearly, A and B are Subspace of $\mathbb{R}^3$

for $(1,0,0)$ and $(0,1,0) \in A \cup B$

then, $(1,0,0) + (0,1,0) = (1,1,0) \notin A \cup B$.

$\therefore$ $A \cup B$ is not a Subspace of $\mathbb{R}^3$.

Theorem 5 :-

P.T, the union of two Subspace of a Vector Space is a Subspace iff One is Contained in other.

Proof:-

Let A and B be a 2 Subspaces of V

$\ni$ $A \subseteq B$ (or) $B \subseteq A$.

To prove, $A \cup B$ is a subspace of V.

$\therefore$ $A \subseteq B$; $A \cup B = B$

& $B \subseteq A$; $A \cup B = A$

$\therefore$ $A \cup B$ is a subspace of V.

Conversely,

$A \cup B$ is a subspace of V

We have to prove,

$A \subseteq B$ (or) $B \subseteq A$

Suppose, $A \not\subseteq B$ (or) $B \not\subseteq A$

$\exists$ x, y $\ni$

$x \in A$ and $x \notin B$ —— (1)

$y \in B$ and $y \notin A$. —— (2)

Clearly $x, y \in A \cup B$ and $A \cup B$ is a subspace of V

$\Rightarrow$ $x+y \in A \cup B$

Hence, $x+y \in A$ & $x+y \in B$.

Case 1

If $x+y \in A$

$\therefore x \in A$ and $x+y \in A$

$\Rightarrow -x \in A$ ($\because A$ is a Subspace of V)

$\therefore x+y-x \in A$

$\Rightarrow y \in A$

Which is a contradiction to (2)

<u>Case 2</u>

If $x+y \in B$

$\therefore y \in B$ and $x+y \in B$

$\Rightarrow -y \in B$ ($\because B$ is a subspace of V)

$\therefore x+y-y \in B$

$\Rightarrow x \in B$

Which is a contradiction to (1).

$\therefore A \subseteq B$ (or) $B \subseteq A$.

<u>Problems</u> :-

1) S.T, $\{(a_1, a_2, a_3) / a_1 + a_2 = 0\}$ is a Subspace of $V_3(R)$

<u>Soln</u>

Let $W = \{(a_1, a_2, a_3) / a_1 + a_2 = 0\}$

and $x, y \in W$

then, $x = (a_1, a_2, a_3)$

$\ni a_1 + a_2 = 0$ —— (1)

and $y = (b_1, b_2, b_3)$

$\ni b_1 + b_2 = 0$. —— (2)

Consider, $\alpha, \beta \in R$

$\alpha x + \beta y = \alpha (a_1, a_2, a_3) + \beta (b_1, b_2, b_3)$

$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\beta b_1, \beta b_2, \beta b_3)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$

We have to prove,

$\alpha a_1 + \beta b_1 + \alpha a_2 + \beta b_2 = 0.$

L.H.S,

$\alpha a_1 + \beta b_1 + \alpha a_2 + \beta b_2 = \alpha(a_1 + a_2) + \beta(b_1 + b_2)$

$= \alpha(0) + \beta(0)$ (by ① & ②).

$= 0.$

$\Rightarrow \alpha x + \beta y \in W$

$\Rightarrow$ W is a Subspace of $V_3(R)$.

2) Check $\{(a_1, a_2, a_3) / a_1^2 + a_2^2 = a_3^2\}$ is a Subspace of $V_3(R)$

Soln

Let $W = \{(a_1, a_2, a_3) / a_1^2 + a_2^2 = a_3^2\}$

and let $x, y \in W$

then, $x = (a_1, a_2, a_3)$ | $y = (b_1, b_2, b_3)$

$\Rightarrow a_1^2 + a_2^2 = a_3^2$ ① | $\Rightarrow b_1^2 + b_2^2 = b_3^2$ ②

Consider $\alpha, \beta \in R$

$\alpha x + \beta y = \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$

We have to prove.

$(\alpha a_1 + \beta b_1)^2 + (\alpha a_2 + \beta b_2)^2 = (\alpha a_3 + \beta b_3)^2$

Consider,

L.H.S $= (\alpha a_1 + \beta b_1)^2 + (\alpha a_2 + \beta b_2)^2$

$= \alpha^2 a_1^2 + \beta^2 b_1^2 + 2\alpha\beta a_1 b_1 + \alpha^2 a_2^2 + \beta^2 b_2^2 + 2\alpha a_2 \beta b_2$

$$= \alpha^2(a_1^2+a_2^2) + \beta^2(b_1^2+b_2^2)$$
$$+ 2\alpha\beta(a_1a_2+b_1b_2)$$

$$= \alpha^2(a_3^2) + \beta^2 b_3^2 + 2\alpha\beta(a_1a_2+b_1b_2) \quad (\text{by } ① \,\&\, ②)$$

$$\neq (\alpha a_3 + \beta b_3)^2$$

$\therefore$ W is not a Subspace of $V_3(R)$

3) Check $W = \{(a_1, a_2, a_3) \in R^3 / a_1 + 2a_2 - 3a_3 = 1\}$ is a Subspace of $V_3(R)$ or not?

Soln

Let $x, y \in W$

then $x = (a_1, a_2, a_3)$

$\Rightarrow a_1 + 2a_2 - 3a_3 = 1$ —— ①

and $y = (b_1, b_2, b_3)$

$\Rightarrow b_1 + 2b_2 - 3b_3 = 1$ —— ②

Consider, $\alpha, \beta \in R$

$$\alpha x + \beta y = \alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3)$$

$$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3)$$

We have to prove,

$$\alpha a_1 + \beta b_1 + 2(\alpha a_2 + \beta b_2) - 3(\alpha a_3 + \beta b_3) = 1$$

L.H.S

$$\alpha a_1 + \beta b_1 + 2(\alpha a_2 + \beta b_2) - 3(\alpha a_3 + \beta b_3)$$

$$= \alpha a_1 + 2\alpha a_2 - 3\alpha a_3$$
$$+ \beta b_1 + 2\beta b_2 - 3\beta b_3$$

$$= \alpha(a_1 + 2a_2 - 3a_3) + \beta(b_1 + 2b_2 - 3b_3)$$

$$= \alpha(1) + \beta(1) = \alpha + \beta \neq 1$$

$\therefore$ W is not a Subspace of $V_3(R)$

4) Let $V(R)$ be the real Vector Space of all fns from R into R. S.T, the Set W of all fns f $\ni$ $f(x^2) = [f(x)]^2$ does not Constitute a Subspace of $V(R)$

Soln

Given $V(R) = \{f / f : R \to R\}$

and $W = \{f / f : R \to R \text{ and } f(x^2) = [f(x)]^2\}$

Let $f, g \in W$

then $f(x^2) = [f(x)]^2$ — (1)

& $g(x^2) = [g(x)]^2$ — (2)

consider, $\alpha, \beta \in R$

then, To Prove $\alpha f + \beta g \in W$

(i.e) to prove,

$$(\alpha f + \beta g)(x^2) = [(\alpha f + \beta g)(x)]^2$$

L.H.S

$$(\alpha f + \beta g)(x^2) = (\alpha f)(x^2) + (\beta g)(x^2)$$
$$= \alpha f(x^2) + \beta g(x^2)$$
$$= \alpha [f(x)]^2 + \beta [g(x)]^2$$

(from (1) & (2)).

R.H.S

$$[(\alpha f + \beta g)(x)]^2 = [\alpha f(x) + \beta g(x)]^2$$
$$= \alpha^2 [f(x)]^2 + \beta^2 [g(x)]^2 + 2\alpha\beta f(x) g(x)$$

$\Rightarrow$ L.H.S $\neq$ R.H.S.

$\therefore$ W is not a Subspace of $V(R)$.

5) Let $V_n(R)$ be the V.S over the field of real numbers R. Define $W = \{(a_1, a_2, \ldots, a_n) \in R^n : a_i \in R, i = 1, 2, \ldots, n$ and $a_2 = a_1^2\}$.

Check whether $W(R)$ forms a subspace of $V_n(R)$.

Soln

Let $x, y \in W$

then $x = (a_1, a_2, \ldots, a_n)$

$\ni a_2 = a_1^2$ —— (1)

& $y = (b_1, b_2, \ldots, b_n)$

$\ni b_2 = b_1^2$ —— (2)

Consider, $\alpha, \beta \in R$

then, $\alpha x + \beta y = \alpha(a_1, \ldots, a_n) + \beta(b_1, b_2, \ldots, b_n)$

$= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots, \alpha a_n + \beta b_n)$

To prove,

$\alpha a_2 + \beta b_2 = (\alpha a_1 + \beta b_1)^2$

$R.H.S = (\alpha a_1 + \beta b_1)^2$

$= \alpha^2 a_1^2 + \beta^2 b_1^2 + 2\alpha\beta a_1 b_1$

$= \alpha^2 a_2 + \beta^2 b_2 + 2\alpha\beta a_1 b_1$

$\neq \alpha a_2 + \beta b_2$

$\therefore \alpha x + \beta y \notin W$

$\Rightarrow$ W is not a subspace of R.

H.W

1) $\{(a_1, a_2, a_3) / a_1 + a_2 + 2a_3 = 0\}$ is a subspace of $V_3(R)$

2) $\{(a_1, a_2) / a_1, a_2 \in R$ and $a_1^2 = a_2^2\}$ is not a subspace of $V_2(R)$

3) If $V(R)$ be the Vector Space of all 2x2 matrices

Over the real field R. S.T $W = \{A / A^2 = A\}$ is not a Subspace of $V(R)$

4) $V(R)$ be the real Vector fns.

S.T, $W = \{f / f : R \to R$ and $f(0) = f(1)\}$ is a Subspace of $V(R)$

5) $V_n(R)$ be the V.S of real numbers.

S.T, $W = \{(a_1, a_2, \dots, a_n) / a_i \in R, i = 1, 2, \dots, n$ and $a_1 a_2 = 0\}$ is not a Subspace of $V_n(R)$.

Sec 3

Linear Combinations and linear Systems of Equations

Linear Combination:-

Let V be a vector space over a field F. and let $(v_1, v_2, \ldots, v_n) \in V$, then an element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_i \in F$, $i = 1, 2, \ldots, n$ is called linear combination of vectors $v_1, v_2, \ldots, v_n$.

Linear Span :-

Let V be a V.S over a field F and S be a nonempty subset of V (i.e) $S \subset V$. Then the set of all linear combinations of finite sets of elements of S is called linear span of S and is denoted by $L(S)$.

$$\text{(i.e)} \quad L(S) = \{\alpha_1 v_1 + \ldots + \alpha_n v_n \,/\, \alpha_i \in F, v_i \in S, i = 1, 2, \ldots, n\}$$

Note

1) Any element $L(S)$ is of the form $\sum_{i=1}^{n} \alpha_i v_i$, $\alpha_i \in F$, $v_i \in S$

2) If $S = \varphi$ then $L(S) = \{0\}$.

Example

1) Let $e_1 = (1, 0, 0)$; $e_2 = (0, 1, 0)$; $e_3 = (0, 0, 1)$

$$S = \{e_1, e_2, e_3\}$$

then $L(S) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ where $\alpha_1, \alpha_2, \alpha_3 \in F$

$$= \alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1)$$

$$= (\alpha_1, 0, 0) + (0, \alpha_2, 0) + (0, 0, \alpha_3)$$

$$= (\alpha_1, \alpha_2, \alpha_3) \quad \text{where } \alpha_1, \alpha_2, \alpha_3 \in R$$

$$\therefore L(S) = V_3(R)$$

Theorem 1

Let V be a V.S over a field F and S be a non-empty Subset of V. Then,

(i) $L(S)$ is a Subspace of V

(ii) $S \subseteq L(S)$

(iii) $L(S)$ is a Smallest Subspace of V containing S.

Proof :-

(i) Let $V, W \in L(S)$ and $\alpha, \beta \in F$

then, $V = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$; $\alpha_i \in F, v_i \in S$

and $W = \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n$; $\beta_i \in F, w_i \in S$.

Now Consider,

$$\alpha V + \beta W = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n) + \beta(\beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n)$$

$$= (\alpha\alpha_1) v_1 + \alpha\alpha_2 v_2 + \dots + \alpha\alpha_n v_n + \beta\beta_1 w_1 + \beta\beta_2 w_2 + \dots + \beta\beta_n w_n$$

is also a linear Combination of a finite number of elements of S.

(ii) Let $u \in S$

then, $1 \cdot u \in L(S)$

Hence $S \subseteq L(S)$

(iii) To Prove $L(S)$ is a Smallest Subspace of V containing S.

We have to prove, $L(S) \subseteq W$, Where W is any subspace of V $\Rightarrow S \subseteq W$

Let $u \in L(S)$

then $u = \alpha_1 u_1 + \dots + \alpha_n u_n$

Where $u_i \in S$ and $\alpha_i \in F$

$\because$ $S \subseteq W$, $u_i \in W$

$\therefore$ $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \in W \Rightarrow u \in W$

Hence $L(S) \subseteq W$

Spans of V or S generates V

Let S be a subset of a V.S. V. If Span(S) = V (i.e) L(S) = V, then S spans V (or) S generates V.

Problems :-

1) Check in $V_3(R)$, the Vector $(-2,0,3)$ can be expressed as a linear Combination of Vectors $(1,3,0)$ and $(2,4,-1)$

Soln

To check, we have to find (a_1, a_2)

$$\Rightarrow (-2,0,3) = a_1(1,3,0) + a_2(2,4,-1) \quad \text{--- (i)}$$

$$= (a_1+2a_2,\ 3a_1+4a_2,\ -a_2)$$

equating,

$$a_1 + 2a_2 = -2$$

$$3a_1 + 4a_2 = 0$$

$$-a_2 = 3. \Rightarrow \boxed{a_2 = -3}$$

sub in above eqn

$$3a_1 + 4(-3) = 0$$

$$3a_1 = 12$$

$$\boxed{a_1 = 4}$$

Sub in (i)

$$\Rightarrow (-2,0,3) = 4(1,3,0) - 3(2,4,-1)$$

Hence $(-2,0,3)$ can be expressed as a linear combination of $(1,3,0)$ and $(2,4,-1)$.

H.W

Check in $V_3(R)$, Vector $(3,4,1)$ can be expressed as a linear combination of $(1,-2,1)$ and $(-2,-1,1)$
(No soln).

2) Is the Vector $(2,-5,3)$ in the Subspace of $V_3(R)$ Spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$.

Soln

To check, we have to find (a,b,c)

$\ni (2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$ — (1)

equations.

$2 = a + 2b + c$ — (2)

$-5 = -3a - 4b - 5c$ — (3)

$3 = 2a - b + 7c$ — (4)

Solving the above 3 eqns.

(2) $\times 3 \Rightarrow \cancel{3a} + 6b + 3c = 6$

(3) $\times 1 \Rightarrow -\cancel{3a} - 4b - 5c = -5$

$2b - 2c = 1$ — (5)

(3) $\times 2 \Rightarrow -\cancel{6a} - 8b - 10c = -10$.

(4) $\times 3 \Rightarrow \cancel{6a} - 3b + 21c = 9$

$-11b + 11c = -1 \quad \div -1$

$11b - 11c = 1$ — (6)

We cannot solve (5) & (6) ($\because$ It is a inconsistant eqns) $\therefore$ The (2), (3), (4) does not have a Soln.

$\Rightarrow$ (1) does not holds

H.W.

1) Determine the ~~Subspace~~ V.S $V_3(R)$, express the Vector (1,-2,5) as a linear combination of Vectors (1,1,1), (1,2,3) and (2,-1,1) (Yes) (-6,3,2)

2) Let $S = \{(1,4),(0,3)\}$ be a Subset of the V.S $V_2(R)$. S.T $(2,3) \in L(S)$ (Yes) $(2, -5/3)$

3) Check the Polynomial $2x^3-2x^2+12x-6$ is a linear combination of the Polynomial x^3-2x^2-5x-3 and $3x^3-5x^2-4x-9$ in $P_3(R)$

Soln

To check, we have to find (a_1, a_2)

$\ni$ $2x^3-2x^2+12x-6 = a_1(x^3-2x^2-5x-3)+a_2(3x^3-5x^2-4x-9)$ — (1)

equating x^3

$2 = a_1+3a_2$ — (2)

equating x^2

$-2 = -2a_1-5a_2$ — (3)

equating x

$12 = -5a_1-4a_2$ — (4)

equating cons

$-6 = -3a_1-9a_2$ — (5)

(2) $\times 2 \Rightarrow$ $4 = 2a_1+6a_2$

(3) $\times 1 \Rightarrow$ $-2 = -2a_1-5a_2$

$\boxed{2 = a_2}$

sub in (2)

$2 = a_1+6$

$\boxed{a_1 = -4}$

Substituting a_1 and a_2 in (1),

We get

$2x^3-2x^2+12x-6 = -4(x^3-2x^2-5x-3)+2(3x^3-5x^2-4x-9)$

H.W

Check the polynomial $3x^3-2x^2+7x+8$ is a linear combination of x^3-2x^2-5x-3 and $3x^3-5x^2-4x-9$ in $P_3(R)$. Ans No soln.

4) Determine $L(S)$ generated by $S = \{(1,0,-2), (2,3,4)\}$ in $V_3(R)$.

Soln

To find $L(S)$,

for any $x \in L(S)$ and $(a_1, a_2) \in R$

We have

x is linear combination of $(1,0,-2)$ and $(2,3,4)$

$$\therefore \quad x = a_1(1,0,-2) + a_2(2,3,4)$$

$$= (a_1+2a_2, \; 0+3a_2, \; -2a_1+4a_2).$$

$$\therefore \quad L(S) = \{(a_1+2a_2, \; 0+3a_2, \; -2a_1+4a_2) \,/\, a_1, a_2 \in R\}.$$

5) Determine the Sub space of $V_4(R)$ generated by $(2,0,0,1)$ and $(-1,0,1,0)$.

Soln

Let $S = \{(2,0,0,1), (-1,0,1,0)\}$.

and $L(S)$ is the set of all linear combination of Vectors in S.

$\therefore$ $L(S)$ is the subspace of $V_4(R)$ generated by elements of S.

We have to find $L(S)$

for any, $x \in L(S)$ and $(a_1, a_2) \in R$.

$$x = a_1(2,0,0,1) + a_2(-1,0,1,0).$$

$$x = (2a_1 - a_2, \; 0, \; a_2, \; a_1).$$

$\Rightarrow L(S) = \{(2a_1 - a_2, 0, a_2, a_1) \,/\, a_1, a_2 \in R\}$ is the Subspace of $V_4(R)$

6) Find the linear Span of S for $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ in $M_{2\times2}(R)$.

<u>Soln</u>

linear Span of $S = L(S)$

for any, $x \in L(S)$ and $(a_1, a_2) \in R$.

$$x = a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}.$$

$$\therefore L(S) = \left\{ \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix} / a_1, a_2 \in R \right\}$$

<u>H.W</u>

find $L(S)$

1) $S = \{(1,0), (0,1)\}$

2) $S = \{(1,0,0), (2,0,0), (3,0,0)\}$ in $V_3(R)$.

3) Determine the Subspace of $V_3(R)$ generated by $(1,0,0)$ and $(0,3,0)$.

Sec: 4

Linear dependence and Linear Independence :-

Linearly Independent:-

Let V be a V.S over a field F. A finite set of vectors $V_1, V_2, \ldots, V_n \in V$ is said to be linearly independent if

$$\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Note:- For system of eqns, $|A| \neq 0$.

Linearly Dependent:-

Let V be a V.S over a field F. A finite set of vectors $V_1, V_2, \ldots, V_n \in V$ is said to be linearly dependent if

$$\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$$
$$\Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_n \text{ not all zero.}$$

Note:- for system of eqns, $|A| = 0$.

Problems

1) In $V_n(F)$, S.T $\{e_1, e_2, \ldots, e_n)$ is linearly independent set of vectors.

Soln

W.K.T, $e_1 = (1, 0, 0, \ldots, 0)$; $e_2 = (0, 1, 0, \ldots, 0)$
$\ldots$ $e_n = (0, 0, 0, \ldots, 1)$ — (1)

To prove $\{e_1, \ldots, e_n\}$ are linearly independent.

$\therefore$ By defn,

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0. \quad \text{— (2)}$$

We have to prove,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

sub (1) in (2).

$$\therefore \alpha_1(1, 0, \ldots, 0) + \alpha_2(0, 1, 0, \ldots, 0) + \cdots + \alpha_n(0, 0, \ldots, 1) = (0, 0, \ldots, 0)$$

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$$
$$\Rightarrow \alpha_1 = 0 = \alpha_2 = \cdots = \alpha_n = 0.$$

2) In $V_3(R)$, S.T the Vectors $(1,4,-2)$, $(-2,1,3)$ and $(-4,11,5)$ are linearly dependent.

<u>Soln</u>

The Linear Combination

$$\alpha_1(1,4,-2)+\alpha_2(-2,1,3)+\alpha_3(-4,11,5)=(0,0,0)$$

We have to prove,

$\alpha_1, \alpha_2, \alpha_3$ are not all zero.

$$\alpha_1-2\alpha_2-4\alpha_3=0 \quad -(2)$$

$$4\alpha_1+\alpha_2+11\alpha_3=0 \quad -(3)$$

$$-2\alpha_1+3\alpha_2+5\alpha_3=0. \quad -(4)$$

(2), (3) and (4) are inconsistant eqn.

To find $\alpha_1, \alpha_2, \alpha_3$ from (2) & (3)

$$\begin{array}{ccccccc} & \alpha_1 & & \alpha_2 & & \alpha_3 & \\ -2 & & -4 & & 1 & & -2 \\ 1 & & 11 & & 4 & & 1 \end{array}$$

$$\frac{\alpha_1}{-22+4}=\frac{\alpha_2}{-16-11}=\frac{\alpha_3}{1+8}$$

$$\frac{\alpha_1}{-18}=\frac{\alpha_2}{-27}=\frac{\alpha_3}{9}\div\frac{1}{9}$$

$$\alpha_1=-2;\quad \alpha_2=-3,\quad \alpha_3=1$$

$\therefore$ The Vectors are linearly dependent.

3) Let $S=\{(1,-1,2),(2,3,1),(4,5,6)\}$ be a Subset of $V_3(R)$. P.V S is Linearly independent.

<u>Soln</u>

Consider the linear Combination

$$\alpha_1(1,-1,2)+\alpha_2(2,3,1)+\alpha_3(4,5,6)=(0,0,0)$$

$$(\alpha_1 + 2\alpha_2 + 4\alpha_3, -\alpha_1 + 3\alpha_2 + 5\alpha_3, 2\alpha_1 + \alpha_2 + 6\alpha_3) = (0, 0, 0)$$

$$\Rightarrow \quad \alpha_1 + 2\alpha_2 + 4\alpha_3 = 0 \quad \text{--- (2)}$$
$$-\alpha_1 + 3\alpha_2 + 5\alpha_3 = 0 \quad \text{--- (3)}$$
$$2\alpha_1 + \alpha_2 + 6\alpha_3 = 0 \quad \text{--- (4)}$$

Solving above eqns

$1 \times (2) \Rightarrow \cancel{\alpha_1} + 2\alpha_2 + 4\alpha_3 = 0$

$1 \times (3) \Rightarrow \cancel{-\alpha_1} + 3\alpha_2 + 5\alpha_3 = 0$

$5\alpha_2 + 9\alpha_3 = 0$ --- (5)

$2 \times (3) \Rightarrow -2\cancel{\alpha_1} + 6\alpha_2 + 10\alpha_3 = 0$

$1 \times (4) \Rightarrow \cancel{2\alpha_1} + \alpha_2 + 6\alpha_3 = 0$

$7\alpha_2 + 16\alpha_3 = 0$. --- (6)

Solving (5) & (6).

$(5) \times 7 \Rightarrow \quad 35\cancel{\alpha_2} + 63\alpha_3 = 0$

$(6) \times 5 \Rightarrow \quad \cancel{35\alpha_2} + 80\alpha_3 = 0$

$(-) \qquad (-)$

$-17\alpha_3 = 0$

$\boxed{\alpha_3 = 0}$

sub in (5)

$\boxed{\alpha_2 = 0}$

sub α_2, α_3 in (2).

we get $\boxed{\alpha_1 = 0}$

$\therefore$ S is Linearly independent

4) Let $S = \{(1,-2,3,-1), (2,1,-1,2), (3,-1,2,1)\}$ be a subset of $V_4(R)$. Check whether S is linearly independent or not?

Soln

The Linear Combination is

$\alpha_1 (1,-2,3,-1) + \alpha_2 (2,1,-1,2) + \alpha_3 (3,-1,2,1)$
$= (0,0,0,0)$

$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ —— (1)
$-2\alpha_1 + \alpha_2 - \alpha_3 = 0$ —— (2)
$3\alpha_1 - \alpha_2 + 2\alpha_3 = 0$ —— (3)
$-\alpha_1 + 2\alpha_2 + \alpha_3 = 0$ —— (4)

Solve (1) & (2)

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ 1 & -1 & -2 & 1 \end{array}$$

$$\frac{\alpha_1}{-2-3} = \frac{\alpha_2}{-6+1} = \frac{\alpha_3}{1+4}$$

$$\frac{\alpha_1}{-5} = \frac{\alpha_2}{-5} = \frac{\alpha_3}{5} \div \frac{1}{-5}$$

$\Rightarrow \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -1$

Substitute the above values in (3) & (4) and verify the Soln.

$\therefore$ S is Linearly dependent.

5) For What Value of K, the Set $\{(2,-1,3), (3,4,-1), (K,2,1)\}$ becomes linearly independent.

Soln

The linear combination is

$\alpha_1(2,-1,3) + \alpha_2(3,4,-1) + \alpha_3(K,2,1) = 0$.

$\Rightarrow \alpha_1 \neq \alpha_2 \neq \alpha_3 = 0$. (If the set is Linearly independent)

Suppose the given set is linearly dependent.

$2\alpha_1 + 3\alpha_2 + K\alpha_3 = 0$ —— (1)
$-\alpha_1 + 4\alpha_2 + 2\alpha_3 = 0$ —— (2)
$3\alpha_1 - \alpha_2 + \alpha_3 = 0$ —— (3).

then Solving 2 & 3.

$$\begin{matrix} 4 & 2 & -1 & 4 \\ -1 & 1 & 3 & -1 \end{matrix}$$

$$\frac{\alpha_1}{4+2} = \frac{\alpha_2}{6+1} = \frac{\alpha_3}{1-12}$$

$$\frac{\alpha_1}{6} = \frac{\alpha_2}{7} = \frac{\alpha_3}{-11}$$

$\Rightarrow \alpha_1 = 6; \ \alpha_2 = 7; \ \alpha_3 = -11$

If the System is linearly dependent, then above Value should satisfy 1.

$$2(6)+3(7)-11\cancel{\alpha_3}\,K = 0.$$

$$12+21-11\cancel{\alpha_3}\,K = 0$$

$$33-11\cancel{\alpha_3}\,K = 0$$

$$-11\cancel{\alpha_3}\,K = -33$$

$$\boxed{\cancel{\alpha_3}\,K = 3}$$

$\therefore$ for $K=3$, the System is linearly dependent

for $K \neq 3$, the System is linearly independent.

6) Determine Whether the set $S = \{1+x, x+x^2, x^2+1\}$ of $P_2(R)$ is linearly independent or linearly dependent.

Soln

The linear Combination of Vectors of S is

$\alpha_1(1+x) + \alpha_2(x+x^2) + \alpha_3(x^2+1) = 0.$

equation x^2, x, and constant

$\alpha_3 + \alpha_2 = 0. \ — 1$

$\alpha_1 + \alpha_2 = 0. \ — 2$

$\alpha_1 + \alpha_3 = 0. \ — 3.$

from 3, $\alpha_1 = -\alpha_3$.

sub in 1.

$-\cancel{\alpha_1} + \alpha_2 = 0.$

$2 \times 1 \Rightarrow \cancel{\alpha_1} + \alpha_2 = 0$

$2\alpha_2 = 0 \Rightarrow \alpha_2 = 0.$

$\therefore$ S is Linearly independent $\quad \because \alpha_1 = 0, \alpha_3 = 0.$

7) Let $\{u, v, w, z\}$ be a set of linearly independent vectors of a vectors space. Check, whether the set $S = \{u-3z, v+2u, 2v-w, w+z\}$ is linearly independent.

Soln

The linear combination of S is

$$\alpha_1(u-3z) + \alpha_2(v+2u) + \alpha_3(2v-w) + \alpha_4(w+z) = 0.$$

equating the coeff of u, v, w, z

$$\alpha_1 + 2\alpha_2 = 0. \quad \text{---} (1)$$

$$\alpha_2 + 2\alpha_3 = 0 \quad \text{---} (2)$$

$$-\alpha_3 + \alpha_4 = 0. \quad \text{---} (3)$$

$$-3\alpha_1 + \alpha_4 = 0. \quad \text{---} (4)$$

from (1), $\alpha_2 = \frac{-\alpha_1}{2}$

sub in (2)

$$-\frac{\alpha_1}{2} + 2\alpha_3 = 0.$$

$$\cancel{-}2\alpha_3 = \cancel{+}\frac{\alpha_1}{2}$$

$$\alpha_3 = \frac{\alpha_1}{4}$$

sub in (3)

$$-\frac{\alpha_1}{4} + \alpha_4 = 0.$$

$$-\alpha_1 + 4\alpha_4 = 0 \quad \text{---} (5)$$

solving (4) & (5)

$$(5)\times 3 \Rightarrow -3\cancel{\alpha_1} + 12\alpha_4 = 0.$$

$$(4)\times 1 \Rightarrow -\cancel{3}\alpha_1 + \alpha_4 = 0$$

$$(+) \quad (-)$$

$$11\alpha_4 = 0.$$

$$\alpha_4 = 0.$$

sub in (5) $\alpha_1 = 0$.

$\therefore \alpha_2 = \alpha_3 = 0$.

$\therefore S$ is linearly independent

H.W

Check the following S is L.I or L.D.

1) $S = \{(1,1,0), (1,0,1), (0,1,1)$ Ans L.I.

2) $S = \{u-v, v-w, w-u\}$ Ans L.D.

3) $S = \{1, x, x(1,x)\}$ Ans L.I.

4) $S = \{1, x, 1+x+x^2\}$ Ans L.I

5) $S = \{(1,2,0), (0,3,1), (-1,0,1)\}$ Ans L.I

6) $S = \{(1,3,2), (1,-7,-8), (2,1,-1)\}$ Ans L.D.

7) $S = \{(1,2,1), (3,1,5), (3,-4,7)\}$ Ans L.D.

8) $S = \{(2,-2,-4), (1,9,3), (-2,-4,1), (3,7,-1)\}$ Ans L.D.

9) Under What Conditions on the Scalar 'a' are the Vector $(a,1,0)$, $(1,a,1)$ and $(0,1,a)$ in $V_3(R)$ linearly dependent.

Soln

Consider the Linear combination

$$a_1(a,1,0) + a_2(1,a,1) + a_3(0,1,a) = 0$$

$$\Rightarrow \quad aa_1 + a_2 = 0.$$

$$a_1 + aa_2 + a_3 = 0.$$

$$a_2 + aa_3 = 0.$$

Given the Vectors are Linearly dependent.

$$\therefore |A| = 0.$$

$$\Rightarrow \begin{vmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{vmatrix} = 0$$

$$a(a^2-1) - 1(a) = 0$$

$$a^3 - a - a = 0. \Rightarrow a^3 - 2a = 0.$$

$$a(a^2-2)=0$$

$$a=0 \quad \Big| \quad a^2-2=0$$

$$a^2=2$$

$$a=\pm\sqrt{2}$$

Sec 5

Bases and Dimension

Bases

Definition :-

Let V be a Vector Space. A Subset S of V is called basis of V, if

(1) S is Linearly Independent Set

(2) S Spans V (or) S generates V (i.e) $L(S)=V$.

Example

1) φ is L.I and $L(\varphi)=0$

$\therefore$ φ is a basis of zero Vector Space.

2) $S=\{e_1, e_2, \dots, e_n\}$ is a basis of $V_n(F)$

3) $S=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ is a basis of $M_{2\times2}(R)$

4) $S=\{1\}$ is a basis for the V.S R over R.

5) $S=\{1, i\}$ is a basis for the V.S C over R.

6) $S=\{1, x, x^2, \dots, x^n\}$ is a basis for $P_n(F)$

7) $S=\{1, x, x^2, \dots\}$ is a basis for $P(F)$.

Theorem 1

Let V be a V.S over a field F and $S=\{u_1, u_2, \dots, u_n\}$ be a Subset of V. Then S is a basis of V iff every element of V can be expressed as a linear Combination of elements of S.

Proof

Assume $S = \{u_1, u_2, \dots, u_n\}$ is a basis for V.

We have to prove,

every element of V can be uniquely expressed as a linear combination of elements of S.

$\because$ S is a basis of V.

S is Linearly independent and $L(S) = V$

$\therefore$ every element $v \in V$ is of the form

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n.$$

To Prove Uniqueness,

Suppose,

$$a_1u_1 + a_2u_2 + \dots + a_nu_n \neq b_1u_1 + b_2u_2 + \dots + b_nu_n.$$

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \dots + (a_n - b_n)u_n = 0$$

$$\Rightarrow a_1 - b_1 = 0, \ a_2 - b_2 = 0, \dots, a_n - b_n = 0$$

$$\Rightarrow a_i - b_i = 0$$

$$a_i = b_i, \ i = 1, 2, \dots, n.$$

So, $v \in V$ can be uniquely expressed as a linear combination of element of S.

Conversely,

Assume, every element of V can be uniquely expressed as a linear combination of element of S.

To prove,

S is a basis of V.

By assumption, $L(S) = V$.

To prove S is L.I,

Let $a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$.

Also $0u_1 + 0u_2 + \dots + 0u_n = 0$.

Thus, we have expressed 0 as a linear combination of vectors of S in 2 ways

$\therefore$ By hypothesis,

$a_1 = a_2 = \cdots = a_n = 0.$

Hence, S is L.I.

$\Rightarrow$ S is a basis of V.

Problems

1) P.T $S = \{(1,0,0), (0,1,0), (1,1,1)\}$ is a basis of $V_3(R)$

Soln

To prove S is a basis,
We have to prove,
any element $(a,b,c) \in V_3(R)$ can be uniquely expressed as a linear combination of S.

Let
$(a,b,c) = a_1(1,0,0) + a_2(0,1,0) + a_3(1,1,1)$

$a = a_1 + a_3$
$b = a_2 + a_3$
$c = a_3.$

$a = a_1 + c$ | $b = a_2 + c$
$a_1 = a - c$ | $a_2 = b - c$

$\therefore (a,b,c) = (a-c)(1,0,0) + (b-c)(0,1,0) + c(1,1,1).$

$\Rightarrow$ S is a basis of $V_3(R)$

2) Using the definition of basis check $S = \{(1,0,0),(1,1,0)\}$ is a basis or not of V.S $V_3(R)$.

Soln

To prove S is a basis of $V_3(R)$.
We have to prove,

S is L.I and $L(S) = V_3(R)$.

Consider the linear combination,

$$a_1(1,0,0) + a_2(1,1,0) = (0,0,0)$$

$$a_1 + a_2 = 0$$
$$a_2 = 0$$
$$\Rightarrow a_1 = 0.$$

$\therefore$ S is linearly independent

To prove $L(S) = V_3(R)$

Now, $L(S) = \{a_1(1,0,0) + a_2(1,1,0) \,/\, a_1, a_2 \in R\}$

$= \{(a_1 + a_2, a_2, 0) \,/\, a_1, a_2 \in R\}$.

$\neq V_3(R)$

$\Rightarrow$ S does not spans $V_3(R)$.

$\Rightarrow$ S is not a basis of $V_3(R)$.

4) S.T $S = \{(1,0,0), (0,1,0), (1,1,1), (1,1,0)\}$ spans the vector space $V_3(R)$ but is not a basis.

Soln

Let $S' = \{(1,0,0), (0,1,0), (1,1,1)\}$.

then $L(S') = \{a_1(1,0,0) + a_2(0,1,0) + a_3(1,1,1) \,/\, a_1, a_2, a_3 \in R\}$

$= \{(a_1 + a_3, a_2 + a_3, a_3) \,/\, a_1, a_2, a_3 \in R\}$

$\Rightarrow L(S') = V_3(R)$

$\because S \supset S'$, we have $L(S) = V_3(R)$

$\Rightarrow$ S Spans $V_3(R)$.

To check Linearly independent,

Consider the linear combination,

$a_1(1,0,0) + a_2(0,1,0) + a_3(1,1,1) + a_4(1,1,0) = (0,0,0)$

$a_4 + a_1 + a_3 = 0$

$a_4 + a_2 + a_3 = 0.$

$a_3 = 0.$

$a_4 + a_1 = 0$

$a_4 + a_2 = 0.$

$\Rightarrow a_1 = -a_4 = a_2 \; ; \; a_3 = 0.$

$\therefore$ all a_i's are not equal to zero.

$\therefore$ They are linearly dependent.

$\Rightarrow$ S is not a basis of $V_3(R)$

Finite Dimensional Vector Space :-

Defn

Let V be a V.S over a field F. V is said to be finite dimensional, if $\exists$ a finite Subset S of $V \ni L(S)=V$.

A V.S that is not finite dimensional is called infinite - dimensional.

Dimension of V.S

Let V be a finite dimensional V.S over a field F. The number of elements in any basis of V is Called dimension of V and it is denoted by $\dim V$.

Eg

1) The V.S $\{0\}$ has dim 0.
2) $\because \{e_1, e_2, \dots, e_n\}$ is a basis of $V_n(R)$
then, $\dim V_n(R) = n$.
3) $M_{2\times 2}(R)$ is a V.S of dim 4.
$\because \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $M_{2\times 2}(R)$
4) $\{1, x, x^2, \dots, x^n\}$ is a basis for $V = P_n(F)$
then $\dim P_n(F) = n+1$.

Result :-

1) Let V be a V.S having a finite basis. Then, every basis for V Contains the Same number of Vectors.
2) Let V is a V.S of $\dim(n)$.
then (i) Any set of 'm' Vectors, $m>n$ is Lin Dep.
(ii) Any set of 'm' Vector, $m<n$ Cannot Span V

3) Let V be a finite dim V.S over F. Let A and B be the Subspace of V. Then

$$\dim(A+B) = \dim(A) + \dim B) - \dim(A \cap B)$$

Problems:-

1) Let $V = V_2(R)$ and $v_1 = e_1 + e_2 = (1,1)$, $v_2 = e_1 - e_2 = (1,-1)$.
P.T, $S = \{v_1, v_2\}$ is a basis of $V_2(R)$.

Soln

To Prove S is a basis of $V_2(R)$.

(i) S is a linearly independent

(ii) $L(S) = V_2(R)$.

(i) The linear combination is

$$a_1(1,1) + a_2(1,-1) = 0.$$

then $a_1 + a_2 = 0$ — ①

$a_1 - a_2 = 0$. — ②

solving ① & ②
we get
$a_1 = 0, a_2 = 0$.

$\therefore$ S is linearly independent.

(ii) $L(S) = \{a_1(1,1) + a_2(1,-1) / a_1, a_2 \in R\}$

$= \{(a_1+a_2, a_1-a_2) / a_1, a_2 \in R)$

$\therefore$ any element $(a,b) \in V_2(R)$ can be express as

$a = a_1 + a_2$	$a = a_1 + a_2$
$b = a_1 - a_2$	$-b = -a_1 + a_2$
$2a_1 = a+b$	$a-b = 2a_2$
$a_1 = \frac{a+b}{2}$	$a_2 = \frac{a-b}{2}$.

$\therefore$ (a,b) of $V_2(R)$ can be uniquely expressed as

$$(a,b) = \frac{1}{2}(a+b)(1,1) + \frac{1}{2}(a-b)(1,-1)$$

$\Rightarrow L(S) = V_2(R)$. $\therefore$ S is a basis of $V_2(R)$

2) S.T $\{x, 3x^2, 5+x\}$ is a basis of $P_2(R)$.

<u>Soln</u>

To Prove $S = \{x, 3x^2, 5+x\}$ is a basis of $P_2(R)$

We have to prove,

(i) S is Linearly independent

(ii) $L(S) = P_2(R)$

(i) To Prove S is Linearly independent,

Consider the Linear Combination,

$$a_1 x + a_2 3x^2 + a_3(5+x) = 0.$$

equating the coeff of x^2, x, constant

$$3a_2 = 0 \; ; \; a_1 + a_3 = 0, \; 5a_3 = 0$$

$$\Rightarrow a_1 = 0, \; a_2 = a_3 = 0.$$

$\therefore$ S is Linearly independent.

(ii) $L(S) = \{a_1 x + a_2 3x^2 + a_3(5+x) \,/\, a_1, a_2, a_3 \in R\}$

$= \{3a_2 x^2 + (a_1 + a_3)x + 5a_3 \,/\, a_1, a_2, a_3 \in R\}$.

Consider one ~~element~~ polynomial

$ax^2 + bx + c \in P_2(x)$

$\Rightarrow ax^2 + bx + c = 3a_2 x^2 + (a_1 + a_3)x + 5a_3$

$\Rightarrow a = 3a_2$

$\Rightarrow \boxed{a_2 = a/3}$

$c = 5a_3$

$a_3 = c/5$

$a_1 + a_3 = b$.

$a_1 + c/5 = b$

$\boxed{a_1 = b - c/5} = \frac{5b-c}{5}$.

$\exists$ any Polynomial in $P_2(x)$ can be uniqluy expressed as.

$$ax^2 + bx + c = \frac{3a}{3} x^2 + \left(\frac{5b-c}{5} + \frac{c}{5}\right)x + 5(c/5).$$

$\therefore L(S) = P_2(R)$

$\Rightarrow$ S is a basis of $P_2(R)$

3) Examine Whether or not the following Vectors forms a basis of $V_3(R)$, $(1,1,2)$, $(1,2,5)$, $(5,3,4)$.

Soln

Let $S = \{(1,1,2), (1,2,5), (5,3,4)\}$.

To prove S is a basis of $V_3(R)$

we have to prove

(i) S is Linearly independent

(ii) $L(S) = V_3(R)$.

(i) To prove S is Linearly independent,

Consider the Linear Combination

$$a_1(1,1,2) + a_2(1,2,5) + a_3(5,3,4) = 0$$

$$a_1 + a_2 + 5a_3 = 0 \quad \text{--- (1)}$$

$$a_1 + 2a_2 + 3a_3 = 0 \quad \text{--- (2)}$$

$$2a_1 + 5a_2 + 4a_3 = 0. \quad \text{--- (3)}$$

from (1) & (2).

$$\begin{array}{cccc} 1 & 5 & 1 & 1 \\ 2 & 3 & 1 & 2 \end{array}$$

$$\frac{a_1}{3-10} = \frac{a_2}{5-3} = \frac{a_3}{2-1}$$

$$\frac{a_1}{-7} = \frac{a_2}{2} = \frac{a_3}{1}$$

sub in (3)

$$2(-7) + 5(2) + 4(1)$$

$$= -14 + 10 + 4 = 0.$$

$\Rightarrow a_1 = -7;\ a_2 = 2;\ a_3 = 1.$

$\Rightarrow S$ is Linearly dependent.

$\therefore$ S is not a basis of $V_3(R)$.

4) S.T $S = \{1, x, x^2, \ldots, x^n\}$ of $(n+1)$ Polynomial in $P_n(R)$ of all polynomial x is a basis of the V.S in x over a field of real numbers.

Soln

To Prove $S = \{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(R)$

We have to prove
(i) S is Linearly independent
(ii) $L(S) = P_n(R)$

(i) To Prove S is Linearly independent
consider the Linear combination.
$a_1 + a_2x + a_3x^2 + \ldots + a_{n+1}x^n = 0$
equating the coeff of $x^n, \ldots$, constan
$\Rightarrow a_1 = 0, a_2 = 0, \ldots, a_n = 0 = a_{n+1}$
$\Rightarrow S$ is Linearly independent

(ii) $L(S) = \{a_1 + a_2x + a_3x^2 + \ldots + a_{n+1}x^n \,/\, a_1, a_2, \ldots, a_{n+1} \in R\}$

Consider a polynombial.
$b_1 + b_2x + \ldots + b_{n+1}x^n \in P_n(R)$
$\ni \quad b_1 + b_2x + \ldots + b_{n+1}x^n = a_1 + a_2x + \ldots + a_{n+1}x^n$
$\Rightarrow b_1 = a_1;\ b_2 = a_2;\ \ldots;\ b_{n+1} = a_{n+1}$

$\Rightarrow$ any polymial in $P_n(R)$ can be Uniquely expressed as $a_1 + a_2x + \ldots + a_{n+1}x^n$
$\therefore L(S) = P_n(R)$

$\Rightarrow S$ is a basis of $P_n(R)$

5) S.T, the set $\{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ forms a basis for $V_3(C)$ On the field C of complex numbers.

Soln

Let $S = \{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$.
To Prove S is a basis.

We have to prove,

(i) S is Linearly independent

(ii) $L(S) = V_3(C)$.

(i) To prove S is L.I.

Consider the Linear Combination,

$$a_1(1,i,0) + a_2(2i,1,1) + a_3(0,1+i,1-i) = 0$$

$$a_1 + 2a_2 i = 0 \quad \text{--- (1)}$$

$$a_1 i + a_2 + a_3(1+i) = 0 \quad \text{--- (2)}$$

$$a_2 + (1-i)a_3 = 0 \quad \text{--- (3)}$$

$$(2) \times (1-i) \Rightarrow (1-i)i a_1 + (1-i)a_2 + a_3\cancel{(1-i)(1+i)} = 0.$$

$$(3) \times (1+i) \Rightarrow \quad (1+i)a_2 + \cancel{(1+i)(1-i)}a_3 = 0.$$

$$\quad (-) \qquad (-)$$

$$(i+1)a_1 - 2ia_2 = 0 \quad \text{--- (4)}$$

$$(1+i) \times (1) \Rightarrow \cancel{(1+i)}a_1 + 2i(1+i)a_2 = 0$$

$$1 \times (4) \Rightarrow (1+\cancel{i})a_1 - 2ia_2 = 0$$

$$(-) \qquad (+)$$

$$(2i - 2 + 2i)a_2 = 0$$

$$\Rightarrow a_2 = 0.$$

sub in (4) $\Rightarrow a_1 = 0$.

Sub in (3) $\Rightarrow a_3 = 0$.

$\therefore$ S is Linearly Independent.

(ii) $L(S) = \{a_1(1,i,0) + a_2(2i,1,1) + a_3(0,1+i,1-i) / a_1,a_2,a_3 \in \mathbb{C}\}$

$= \{(a_1 + 2a_2 i,\ a_1 i + a_2 + a_3(1+i),\ a_2 + (1-i)a_3) / a_1,a_2,a_3 \in C\}$

Consider a complex numbers.

$(x,y,z) \in C$

$$\ni \quad x = a_1 + 2a_2 i \quad \text{--- (5)}$$

$$y = a_1 i + a_2 + a_3(1+i) \quad \text{--- (6)}$$

$$z = a_2 + (1-i)a_3 \quad \text{--- (7)}$$

$$⑥ \times (1-i) \Rightarrow (1-i)ia_1 + (1-i)a_2 + (1-i)\,a_3(1+i) = (1-i)y$$
$$⑦ \times (1+i) \Rightarrow \underset{(-)}{(1+i)a_2} \qquad \underset{(-)}{+\ (1-i)\,a_3(1+i)} = (1+i)z$$

$$(i+1)a_1 - 2ia_2 = (1-i)y + (1+i)z \quad — ④.$$

$$⑤ \times (1+i) \Rightarrow (1+i)\cancel{a_1} + 2i(1+i)a_2 = (1+i)x$$
$$④ \times 1 \Rightarrow \underset{(-)}{(1+i)}\,a_1 \ \underset{(+)}{-\ 2ia_2} = \underset{(-)}{(1-i)y} + \underset{(-)}{(1+i)z}$$

$$(2i-2+2i)a_2 = (1+i)x + (i-1)y - (1+i)z$$

$$a_2 = \frac{(1+i)x + (i-1)y - (1+i)z}{4i-2}$$

sub a_2 in ①

$$a_1 = \frac{+\cancel{2}i((1+i)x + (i-1)y - (1+i)z}{+\cancel{2}(1-2i)}$$

sub a_2 in ③.

$$a_3 = \frac{(1+i)x + (i-1)y - (1+i)z}{(2-4i)\ (1-i)}.$$

By substituting the above values, we get, any complex number $(x, y, z) \in C$ can be uniquely expressed as the Linear combination in C.

$$\therefore\ L(S) = V_3(C)$$

$\Rightarrow$ S is a basis of $V_3(C)$.

H.W

1) $S\{(1,1,0),(0,1,1),(1,0,1)\}$ is a basis of $V_3(R)$

2) $S=\{(2,-3,1),(0,1,2),(1,1,2)\}$ is a basis of $V_3(R)$

3) $S=\{(2,1,0),(1,-1,0),(4,2,0\}$ is a basis of $V_3(R)$

4) S.T $\{1,i\}$ forms a basis of $C(R)$.