unit - II

Infinite Impulse Response Filters

characteristics of practical frequency selective filters, characteristics of commonly used analog filters - Butterworth filters, chebyshev filters, Design of IIR filters from analog filters (LPF, HPF, BPF, BRF) - Approximation of derivatives, Impulse invariance method, Bilinear transformation. Frequency transformation is the analog domain, Structure of IIR filter - direct form ~~I~~, direct form - II, cascade, Parallel realizations.

IIR - Infinite Impulse Response Filter.

Design of IIR filters:-

1) Approximation of Derivatives.

$$S \rightarrow \frac{1-z^{-1}}{T}$$. It is also called as Backward difference.

2) Bilinear Transformation (BLT)

$$S \rightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

steps to design digital filter using BLT:

1) Find Prewarping analog frequencies

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

2) Using the analog frequencies find $H(s)$ of the analog filter.

3) select the Sampling rate of the digital filter call it T seconds per sample.

4) Substitute $S = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$ into the transfer function found in step 2.

Ex. 2.1. Apply BLT to $H(S) = \frac{2}{(S+1)(S+2)}$ with $T = 1$ sec and find $H(z)$.

$$H(S) = \frac{2}{(S+1)(S+2)}$$

$$H(z) = H(S) \Big/_{S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{2}{(S+1)(S+2)} \Big/_{S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$T = 1$ sec,

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)} = \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

3) Impulse Invariant Method (or) Technique (IIM (or) IIT)

Formula:-

$$\frac{1}{s+a} \Rightarrow \frac{1}{1-e^{-aT}z^{-1}}$$

$$\frac{1}{s-a} \Rightarrow \frac{1}{1-e^{aT}z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \Rightarrow \frac{1-e^{-aT}\cos(aT)\,z^{-1}}{1-2e^{-aT}\cos(bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \Rightarrow \frac{e^{-aT}\sin(bT)\,z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

EX.2.2 Using IIT $H(s)=\frac{1}{(s+1)(s+2)}$ convert analog transformation into digital transformation.

$$H(s)=\frac{1}{(s+1)(s+2)}$$

Apply Partial fraction,

$$H(s)=\frac{A}{s+1}+\frac{B}{s+2}=\frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$$\Rightarrow \quad 1=A(s+2)+B(s+1)$$

Put $s=-1$, $\quad s=-2$

$1=A(-1+2)$ $\quad 1=B(-2+1)$

$\boxed{1=A}$ $\quad 1=B(-1)$

$\boxed{B=-1}$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Using IIT,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-2}z^{-1}}$$

$$= \frac{1}{1-0.367z^{-1}} - \frac{1}{1-0.135z^{-1}}$$

$$= \frac{(1-0.135z^{-1}) - (1-0.367z^{-1})}{(1-0.367z^{-1})(1-0.135z^{-1})}$$

$$= \frac{1-0.135z^{-1}-1+0.367z^{-1}}{1-0.135z^{-1}-0.367z^{-1}+0.}$$

$$H(z) = \frac{0.232z^{-1}}{1-0.502z^{-1}+0.049z^{-2}}$$

Ex. 2.3. Using backward difference method $H(s) = \frac{1}{(s+1)^2}$ and $T = 0.01$ sec.

$T = 0.01$ sec

$$s = \frac{1-z^{-1}}{T}$$

$$H(z) = \frac{1}{\left[\left(\frac{1-z^{-1}}{T}\right)+1\right]^2} = \frac{1}{\left(\frac{1-z^{-1}}{0.01}\right)^2+1+2\left(\frac{1-z^{-1}}{0.01}\right)}$$

$$H(z) = \frac{1}{\frac{1+z^{-2}-2z^{-1}+1+200(1-z^{-1})}{1\times10^{-4}}}$$

$$= \frac{1}{\frac{1+z^{-2}-2z^{-1}+(1\times10^{-4})+(0.02-0.02z^{-1})}{0.0001}}$$

$$H(z) = \frac{0.0001}{z^{-2}-202z^{-1}+1.0201}$$

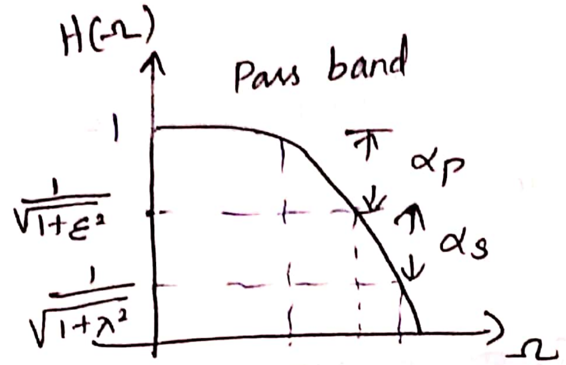
Butterworth Filter :-

* First analog butterworth filter transfer function is determined using specification.
* Analog transfer function is converted into digital filter function using
  - → Bilinear transformation (BLT)
  - → Impulse Invariant technique (IIT)
* The magnitude response of analog Butterworth Low pass filter is,

$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Ω → Normal frequency

Ω_c → Cut-off frequency



From the magnitude response determine $\epsilon, \lambda, \omega_s$ & ω_p.

* using digital transformation, for bilinear transformation,

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$

* For impulse invariant technique,

$$\Omega_p = \frac{\omega_p}{T}$$

$$\Omega_s = \frac{\omega_s}{T}$$

* Determine order of filter.

$$N \geq \frac{\log(\lambda/\epsilon)}{\log[\Omega_s/\Omega_p]}$$

$$\epsilon = \sqrt{10^{0.1\alpha_p} - 1} \qquad \lambda = \sqrt{10^{0.1\alpha_s} - 1}$$

Determine cut-off frequency Ω_c

$$\Omega_c = \frac{\Omega_p}{(\epsilon)^{1/2N}}$$

* Transfer function $H_a(s)$ for the cut-off frequency is obtained by substituting

$s \rightarrow \frac{s}{\Omega}$ for low pass filter

$$S \rightarrow \frac{\Omega_c}{S} \Rightarrow \text{for high pass filter.}$$

* Convert analog transformation into digital transformation using Bilinear transformation and impulse invariant technique.

$N=1$, $\quad H_a(S) = \dfrac{1}{S+1}$

$N=2$, $\quad H_a(S) = \dfrac{1}{S^2+\sqrt{2}\,S+1}$

$N=3$, $\quad H_a(S) = \dfrac{1}{(S+1)(S^2+S+1)}$

$N=4$, $\quad H_a(S) = \dfrac{1}{(S^2+0.76537S+1)(S^2+1.847S+1)}$

EX.2.4. Design an analog Butterworth filter that has a $-2dB$ passband attenuation at a frequency of 20 rad/sec and atleast $-10\,dB$ stopband attenuation at 30 rad/sec.

Given :- $\alpha_p = 2dB$ $\qquad \Omega_p = 20$ rad/sec

$\alpha_s = 10dB$ $\qquad \Omega_s = 30$ rad/sec

$$N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log\sqrt{\dfrac{10-1}{10^{0.2}-1}}}{\log\dfrac{30}{20}}$$

$$\geq 3.37$$

Rounding off N to the next highest integer,

$N = 4$

The normalised lowpass Butterworth filter for $N=4$ is,

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.2}-1)^{1/8}}$$

$$\Omega_c = 21.3868$$

The transfer function for $\Omega_c = 21.3868$ is obtaind by,

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

$$\text{(i.e.) } H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right)+1}$$

$$\times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right)+1}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

EX.Q 5. Design a Butterworth filter using impulse invariance method for the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

Given :-
$$\frac{1}{\sqrt{1+\epsilon^2}} = 0.8$$

$$\Rightarrow \epsilon = 0.75$$

Also
$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\Rightarrow \lambda = 4.899$$

$\omega_s = 0.6\pi$ rad $\qquad \omega_p = 0.2\pi$ rad

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\log \lambda/\epsilon}{\log 1/k} = \frac{\log\left(\frac{4.899}{0.75}\right)}{\log 3} = 1.71$$

Approximating the nearest higher value,

$$N = 2$$

For $N=2$, the transfer function of normalized Butterworth filter is,
$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

$$\Omega_c = \frac{\Omega_p}{(\epsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.231\pi$$

$$H_a(s) = H(s) \Big/_{s \to \frac{s}{0.231\pi}}$$

$$= \frac{0.5266}{s^2 + 1.03s + 0.5266}$$

$$= \frac{0.516j}{s+0.51+j0.51} - \frac{0.516j}{s+0.51-j0.51}$$

$$= \frac{0.516j}{s-(-0.51-j0.51)} - \frac{0.516j}{s-(-0.51+j0.51)}$$

$$H(z) = \frac{0.516j}{1-e^{-0.51\pi}e^{-j0.51\pi}z^{-1}} - \frac{0.516j}{1-e^{-0.51\pi}e^{j0.51\pi}z^{-1}}$$

$(T = 1\text{ sec})$

$$H(z) = \frac{0.3019z^{-1}}{1-1.048z^{-1}+0.36z^{-2}}.$$

Chebyshev filters :-

1. Type I chebyshev filter :-

→ They are all-pole filters that exhibit equiripple behaviour in the pass band and a monotonic characteristics in the stopband.

Type -II chebyshev filter :-

→ It contains both poles and zeros and exhibits a monotonic behaviour in the pass band and an equiripple behaviour in the stop band.

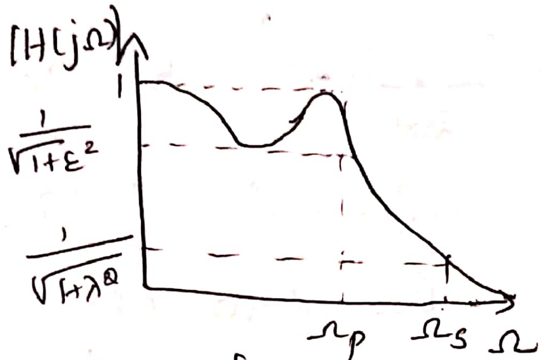


fig:- Type - I

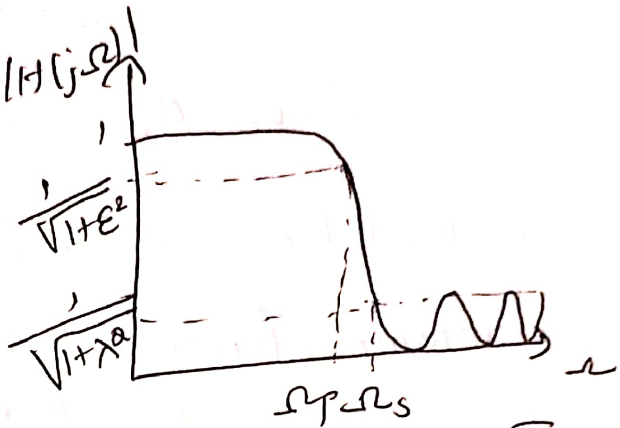


fig: Type -II

Steps to design analog chebyshev Lowpass filter.

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Find the values of a and b, which are minor and major axis of the ellipse.

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} \; ; \; b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where $\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

Ω_p - passband frequency

α_p - Maximum allowable attenuation in the passband.

($\because$ For normalized chebyshev filter $\Omega_p = 1$ rad/sec)

4. Calculate the poles of chebyshev filter which lie on an ellipse by using the formula,

$$s_k = a\cos\phi_k + jb\sin\phi_k \quad , \quad k = 1, 2, \ldots N$$

$$\text{where } \phi_k = \pi/2 + \left(\frac{2k-1}{2N}\right)\pi \quad , \quad k = 1, 2, \ldots N$$

5. Find the denominator polynomial of the transfer function using the above poles.

6. The numerator of the transfer function depends on the value of N.

(a) For N odd substitute $s = 0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function.

($\because$ For N odd the magnitude response $|H(j\Omega)|$ starts at 1).

(b) For N even substitute $s = 0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

Ex. 2.6. Design a chebyshev filter with a maximum passband attenuation of 2.5 dB at $\Omega_p = 20$ rad/sec. and the stop band attenuation of 30 dB at $\Omega_s = 50$ rad/sec.

Given :-

$\Omega_p = 20$ rad/sec ; $\alpha_p = 2.5$ dB

$\Omega_s = 50$ rad/sec ; $\alpha_s = 30$ dB

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(1/k)}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1}$$

$$\lambda = 31.607$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$$\varepsilon = 0.882$$

$$k = \frac{\Omega_p}{\Omega_s} = 0.4$$

$$\therefore N \geq \frac{\cosh^{-1}\left(\frac{31.607}{0.882}\right)}{\cosh^{-1}\left(\frac{1}{0.4}\right)} = 2.726$$

ie) $N = 3$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 2.65$$

$$a = \Omega_p \frac{\left[\mu^{1/N} - \mu^{-1/N}\right]}{2} = 6.6$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 21.06$$

$$S_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2, 3.$$

$$\phi_k = \pi/2 + \left(\frac{2k-1}{2N}\right)\pi \quad ; \quad k = 1, 2, 3$$

$$\phi_1 = 120^\circ, \; \phi_2 = 180^\circ, \; \phi_3 = 240^\circ$$

$$S_1 = -3.3 + j18.23$$

$$S_2 = -6.6$$

$$S_3 = -3.3 - j18.23$$

Denominator of $H(s) = (s+6.6)(s^2+6.6s+343.2)$

Numerator of $H(s) = (6.6)(343.2) = 2265.27$

$$\text{Transfer function } H(s) = \frac{2265.27}{(s+6.6)(s^2+6.6s+343.2)}$$

Ex. 2.7. Design a digital chebyshev filter to meet the constraints

$$\frac{1}{\sqrt{2}} \le H(e^{j\omega}) \le 1 \qquad \text{for } 0 \le \omega \le 0.2\pi$$

$$0 \le |H(e^{j\omega})| \le 0.1 \qquad \text{for } 0.5\pi \le \omega \le \pi$$

by using bilinear transformation and sampling period $T = 1$ sec.

Given $\omega_s = 0.5\pi$; $\omega_p = 0.2\pi$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1 \Rightarrow \lambda = 9.95$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}} \Rightarrow \varepsilon = 1$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan(0.1\pi) = 0.65$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan(\pi/4) = 2$$

$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}}$$

$$N = \frac{\cosh^{-1} 9.95}{\cosh^{-1}\left(\frac{2}{0.65}\right)} = 1.669$$

Approximate $N = 2$,

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1 + \sqrt{2} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$a = 0.65\left[\frac{2.414^{1/2} - 2.414^{-1/2}}{2}\right] = 0.295$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.65\left[\frac{2.414^{1/2} + 2.414^{-1/2}}{2}\right] = 0.717$$

$$\phi_k = \pi/2 + \frac{(2k-1)\pi}{2N} \qquad k=1,2;\ \phi_1 = 135^\circ;$$
$$\phi_2 = 225^\circ$$

$$S_k = a\cos\phi_k + jb\sin\phi_k$$

$$S_1 = 0.295\cos135^\circ + j0.717\sin135^\circ$$
$$S_1 = -0.2086 + j0.507$$
$$S_2 = 0.295\cos225^\circ + j0.717\sin225^\circ$$
$$S_2 = -0.2086 - j0.507$$

Denominator of $H(s)$ is,

$$= (s+0.2086)^2 + (0.507)^2$$

$$\text{Denomin}[H(s)] = s^2 + 0.4172s + 0.3$$

for N, even, Numerator of $H(s)$ is,

$$= \frac{0.3}{\sqrt{1+\epsilon^2}} = 0.212$$

$$H(s) = \frac{0.212}{s^2 + 0.4172s + 0.3}$$

using bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

since $T=1$,

$$H(z) = \frac{0.212(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.8344(1-z^{-2}) + 0.3(1+z^{-1})^2}$$

$$= \frac{0.212(1+2z^{-1}+z^{-2})}{4 - 8z^{-1} + 4z^{-2} + 0.8344 - 0.8344z^{-2} + 0.3 + 0.6z^{-1} + 0.3z^{-2}}$$

$$= \frac{0.212(1+2z^{-1}+z^{-2})}{5.1344 - 7.4z^{-1} + 3.4656z^{-2}}$$

$$H(z) = \frac{0.0413(1+z^{-1})^2}{1 - 1.44z^{-1} + 0.675z^{-2}}$$

Realization of Digital IIR filter:-

IIR filter can be realized in many forms

1. Direct form-I realization
2. Direct form-II realization
3. Cascade form realization
4. parallel form realization.

1. Direct form-I realization:-

Ex.2.8. obtain the direct form-I realization for the system described by difference equation. $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$.

let $x(n) + 0.4x(n-1) = w(n)$

then $y(n) = 0.5y(n-1) - 0.25y(n-2) + w(n)$

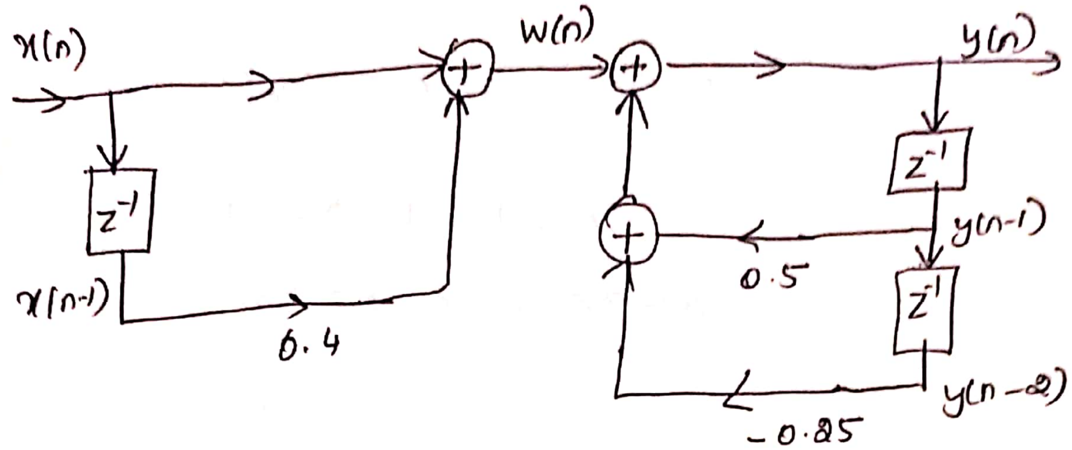


fig: direct form-I realization.

2) Direct form-II.

EX. 2.9. Determine the direct form-II realization for the following system $y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7\, x(n) - 0.252\, x(n-2)$

The system function is given by,

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

let $\dfrac{Y(z)}{W(z)} = 0.7 - 0.252 z^{-2}$

$\Rightarrow Y(z) = 0.7 W(z) - 0.252\, z^{-2} W(z)$

Then, $y(n) = 0.7\, w(n) - 0.252\, w(n-2)$

similarly let $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$

$W(z) = X(z) - 0.1 z^{-1} W(z) + 0.72 z^{-2} W(z)$

then $w(n) = x(n) - 0.1\, w(n-1) + 0.72\, w(n-2)$

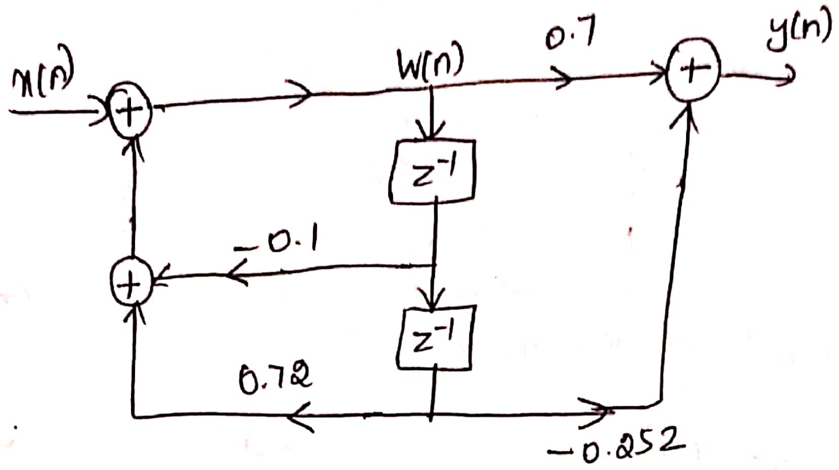


fig: Direct form - II realization.

3. Cascade form.

Ex. 2.10. Realize the system with difference equation $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$ in cascade form.

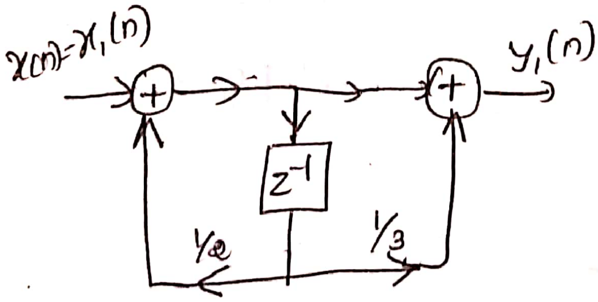
From the difference equation,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

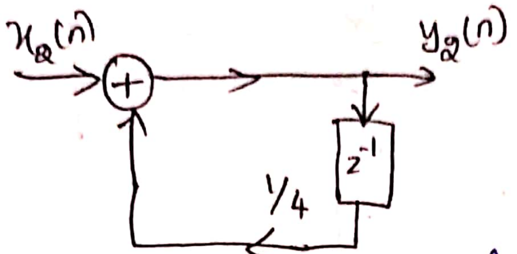
$$H(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} = H_1(z)\,H_2(z)$$

Where $H_1(z) = \dfrac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$ & $H_2(z) = \dfrac{1}{1 - \frac{1}{4}z^{-1}}$

$H_1(z)$ can be realized in direct form II is,



$H_2(z)$ can be realized in direct form-II is,



Cascading the realization of $H_1(z)$ & $H_2(z)$ we have,

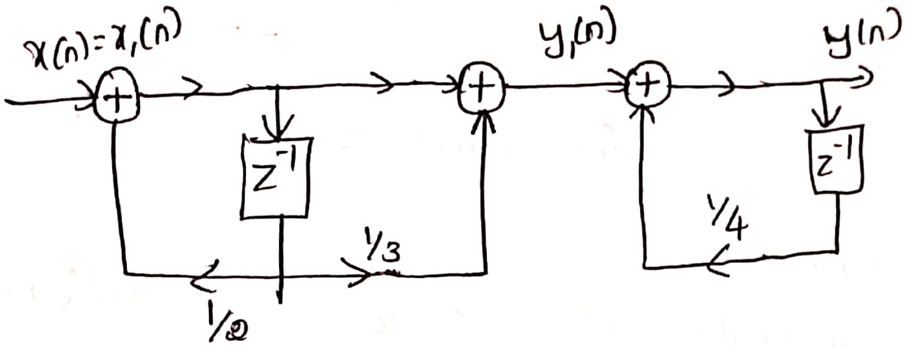


fig: Cascade realization.

4) parallel form :-

EX. 2.11. Realize the system given by difference equation $y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7\, x(n) - 0.252\, x(n-2)$ in parallel form.

The system function of the difference equation is,

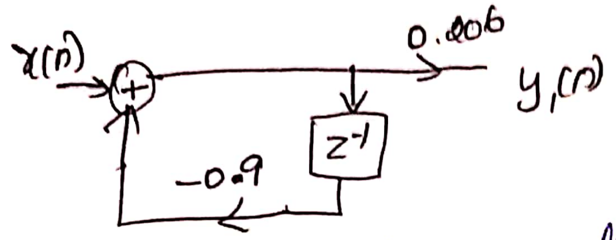
$$H(z) = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$= 0.35 + \frac{0.35 - 0.035 z^{-1}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

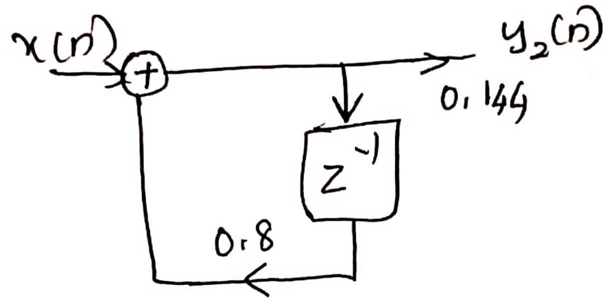
$$= 0.35 + \frac{0.206}{1 + 0.9 z^{-1}} + \frac{0.144}{1 - 0.8 z^{-1}}$$

$$= c + H_1(z) + H_2(z)$$

$H_1(z)$ can be realized in direct form-II as.



$H_2(z)$ can be realized in direct form-II as,



The realization of $H(z)$ is,

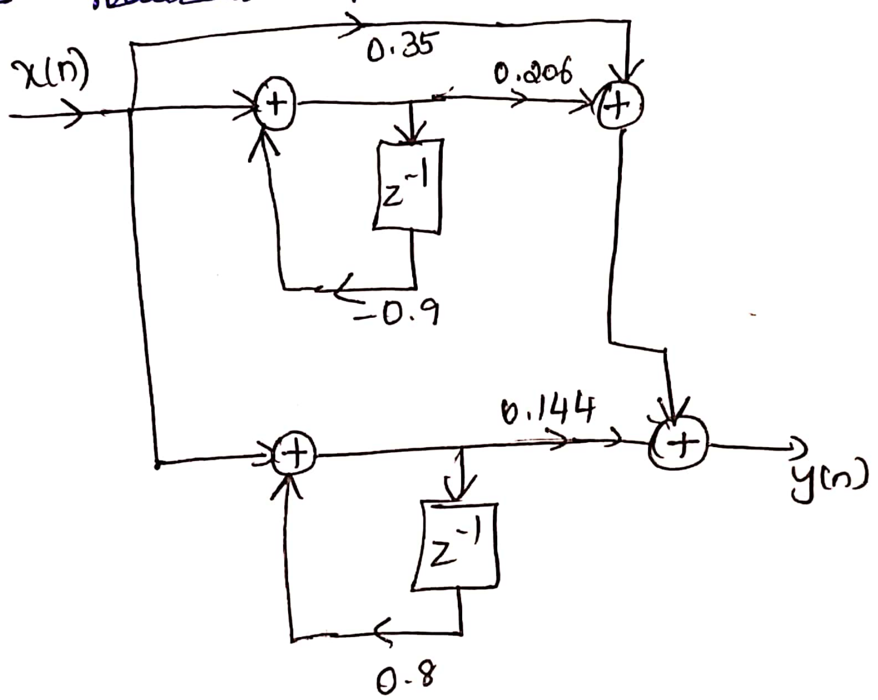


fig: Parallel form realization.