# JEPPIAAR INSTITUTE OF TECHNOLOGY

**"Self-Belief | Self Discipline | Self Respect"**

**DEPARTMENT**

**OF**

**COMPUTER SCIENCE AND ENGINEERING**

**LECTURE NOTES-MA8402**

**PROBABILITY AND QUEUING THEORY**

**(Regulation 2017)**

**Unit I**

1 Introduction

2 Discrete Random Variables

3 Continuous Random Variables

4 Moments

5 Moment generating functions

6 Binomial distribution

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8 Geometric distribution

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11 Gamma distribution

## Introduction

Consider an experiment of throwing a coin twice. The outcomes {HH, HT, TH, TT} consider the sample space. Each of these outcome can be associated with a number by specifying a rule of association with a number by specifying a rule of association (eg. The number of heads). Such a rule of association is called a random variable. We denote a random variable by the capital letter (X, Y, etc) and any particular value of the random variable by x and y.

Thus a random variable X can be considered as a function that maps all elements in the sample space S into points on the real line. The notation X(S)=x means that x is the value associated with the outcomes S by the Random variable X.

## 1 SAMPLE SPACE

Consider an experiment of throwing a coin twice. The outcomes S = {HH, HT, TH, TT} constitute the sample space.

## 2 RANDOM VARIABLE

In this sample space each of these outcomes can be associated with a number by specifying a rule of association. Such a rule of association is called a random variables.

Eg : Number of heads

We denote random variable by the letter (X, Y, etc) and any particular value of the random variable by x or y.

S = {HH, HT, TH, TT} X(S) = {2, 1, 1, 0}

Thus a random X can be the considered as a fun. That maps all elements in the sample space S into points on the real line. The notation X(S) = x means that x is the value associated with outcome s by the R.V.X.

**Example**

In the experiment of throwing a coin twice the sample space S is S = {HH,HT,TH,TT}. Let X be a random variable chosen such that X(S) = x (the number of heads).

**Note**

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

### 2.1 DISCRETE RANDOM VARIABLE

**Definition :** A discrete random variable is a R.V.X whose possible values consitute finite set of values or countably infinite set of values.

**Examples**

All the R.V.'s from Example : 1 are discrete R.V's

**Remark**

The meaning of $P(X \leq a)$.

$P(X \leq a)$ is simply the probability of the set of outcomes 'S' in the sample space for which $X(s) \leq a$.

Or

$P(X \leq a) = P\{S : X(S) \leq a\}$

In the above example : 1 we should write

$P(X \leq 1) = P(HH, HT, TH) = 3/4$

Here $P(X \leq 1) = 3/4$ means the probability of the R.V.X (the number of heads) is less than or equal to 1 is $3/4$ .

**Distribution function of the random variable X or cumulative distribution of the random variable X**

**Def :**

The distribution function of a random variable X defined in $(-\infty, \infty)$ is given by $F(x) = P(X \leq x) = P\{s : X(s) \leq x\}$

**Note**

Let the random variable X takes values $x1, x2, \ldots, xn$ with probabilities $P1, P2, \ldots, Pn$ and let $x1 < x2 < \ldots < xn$

Then we have

$F(x) = P(X < x1) = 0, -\infty < x < x,$

$F(x) = P(X < x1) = 0, P(X < x1) + P(X = x1) \quad = 0 + p1 = p1$

$F(x) = P(X < x2) = 0, P(X < x1) + P(X = x1) + P(X = x2) = p1 + p2$

$F(x) = P(X < xn) = P(X < x1) + P(X = x1) + \ldots + P(X = xn)$

$= p1 + p2 + \ldots + pn \quad = 1$

## 2.2 PROPERTIES OF DISTRIBUTION FUNCTIONS

Property : 1 $P(a < X \le b) = F(b) - F(a)$, where $F(x) = P(X \le x)$

Property : 2 $P(a \le X \le b) = P(X = a) + F(b) - F(a)$

Property : 3 $P(a < X < b) = P(a < X \le b) - P(X = b)$

$= F(b) - F(a) - P(X = b)$ by prob (1)

## 2.3 PROBABILITY MASS FUNCTION (OR) PROBABILITY FUNCTION

Let X be a one dimenstional discrete R.V. which takes the values x1, x2, …… To each possible outcome '$x_i$' we can associate a number $p_i$.

i.e., $P(X = x_i) = P(x_i) = pi$ called the probability of $x_i$. The number $p_i = P(x_i)$ satisfies the following conditions.

(i) $p(x_i) \ge 0, \forall_i$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function p(x) satisfying the above two conditions is called the probability mass function (or) probability distribution of the R.V.X. The probability distribution {xi, pi} can be displayed in the form of table as shown below.

| $X = x_i$ | $x_1$ | $x_2$ | ……. | $x_i$ |
|---|---|---|---|---|
| $P(X = x_i) = p_i$ | $p_1$ | $p_2$ | ……. | $p_i$ |

**Notation**

Let 'S' be a sample space. The set of all outcomes 'S' in S such that X(S) = x is denoted by writing X = x.

$P(X = x) = P\{S : X(s) = x\}$

|||ly $P(x \le a) = P\{S : X() \in (-\infty, a)\}$

and $P(a < x \le b) = P\{s : X(s) \in (a, b)\}$

$P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\}$

$P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}$ and so on.

**Theorem :1** If X1 and X2 are random variable and K is a constant then $KX_1$, $X_1 + X_2$,

$X_1X_2$, $K_1X_1 + K_2X_2$, $X_1$-$X_2$ are also random variables.

**Theorem :2**

If 'X' is a random variable and f(•) is a continuous function, then f(X) is a random variable.

**Note**

If F(x) is the distribution function of one dimensional random variable then

I. $0 \le F(x) \le 1$

II. If $x < y$, then $F(x) \le F(y)$

III. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

IV. $F(\infty) = \lim_{x \to \infty} F(x) = 1$

V. If 'X' is a discrete R.V. taking values $x_1, x_2, x_3$
Where $x_1 < x_2 < x_{i-1}\ x_i$ ……. then
$P(X = x_i) = F(x_i) - F(x_{i-1})$

**Example:**

A random variable X has the following probability function

| Values of X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| Probability P(X) | a | 3a | 5a | 7a | 9a | 11a | 13a | 15a | 17a |

(i) Determine the value of 'a'

(ii) Find $P(X<3)$, $P(X\ge3)$, $P(0<X<5)$

(iii) Find the distribution function of X.

**Solution**

Table 1

| Values of X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| p(x) | a | 3a | 5a | 7a | 9a | 11a | 13a | 15a | 17a |

(i) We know that if p(x) is the probability of mass function then

$$\sum_{i=0}^{8} p(x_i) = 1$$

$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + p(8) = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$81a = 1$

$a = 1/81$

put a = 1/81 in table 1, e get table 2

**Table 2**

| X = x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| P(x) | 1/81 | 3/81 | 5/81 | 7/81 | 9/81 | 11/81 | 13/81 | 15/81 | 17/81 |

(ii) $P(X < 3) = p(0) + p(1) + p(2)$

$= 1/81 + 3/81 + 5/81 = 9/81$

(ii) $P(X \geq 3) = 1 - p(X < 3)$

$= 1 - 9/81 = 72/81$

(iii) $P(0 < x < 5) = p(1) + p(2) + p(3) + p(4)$ here 0 & 5 are not include

$= 3/81 + 5/81 + 7/81 + 9/81$

$$= \frac{3+5+7+8+9}{81} = \frac{24}{81}$$

(iv) To find the distribution function of X using table 2, we get

| **X = x** | **F(X) = P(x ≤ x)** |
|---|---|
| 0 | $F(0) = p(0) = 1/81$ |
| 1 | $F(1) = P(X \leq 1) = p(0) + p(1)$<br>$= 1/81 + 3/81 = 4/81$ |
| 2 | $F(2) = P(X \leq 2) = p(0) + p(1) + p(2)$<br>$= 4/81 + 5/81 = 9/81$ |
| 3 | $F(3) = P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$<br>$= 9/81 + 7/81 = 16/81$ |
| 4 | $F(4) = P(X \leq 4) = p(0) + p(1) + \dots + p(4)$<br>$= 16/81 + 9/81 = 25/81$ |
| 5 | $F(5) = P(X \leq 5) = p(0) + p(1) + \dots + p(4) + p(5)$<br>$= 2/81 + 11/81 = 36/81$ |
| 6 | $F(6) = P(X \leq 6) = p(0) + p(1) + \dots + p(6)$<br>$= 36/81 + 13/81 = 49/81$ |
| 7 | $F(7) = P(X \leq 7) = p(0) + p(1) + \dots + p(6) + p(7)$<br>$= 49/81 + 15/81 = 64/81$ |
| 8 | $F(8) = P(X \leq 8) = p(0) + p(1) + \dots + p(6) + p(7) + p(8)$<br>$= 64/81 + 17/81 = 81/81 = 1$ |

# 3 CONTINUOUS RANDOM VARIABLE

**Def :** A R.V.'X' which takes all possible values in a given internal is called a continuous random variable.

**Example :** Age, height, weight are continuous R.V.'s.

## 3.1 PROBABILITY DENSITY FUNCTION

Consider a continuous R.V. 'X' specified on a certain interval (a, b) (which can also be a infinite interval (-∞, ∞)).

If there is a function y = f(x) such that

$$\lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x)}{\Delta x} = f(x)$$

Then this function f(x) is termed as the probability density function (or) simply density function of the R.V. 'X'.

It is also called the frequency function, distribution density or the probability density function.

The curve y = f(x) is called the probability curve of the distribution curve.

**Remark**

If f(x) is p.d.f of the R.V.X then the probability that a value of the R.V. X will fall in some interval (a, b) is equal to the definite integral of the function f(x) a to b.

$$P(a < x < b) = \int_a^b f(x)dx \quad \text{(or)}$$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

## 3.2 PROPERTIES OF P.D.F

The p.d.f f(x) of a R.V.X has the following properties

1. In the case of discrete R.V. the probability at a point say at x = c is not zero. But in the case of a continuous R.V.X the probability at a point is always zero.

$$P(X = c) = \int_{-\infty}^{\infty} f(x)dx = [x]_c^C = C - C = 0$$

2. If x is a continuous R.V. then we have p(a ≤ X ≤ b) = p(a ≤ X < b)

   = p(a < X V b)

**IMPORTANT DEFINITIONS INTERMS OF P.D.F**

If f(x) is the p.d.f of a random variable 'X' which is defined in the interval (a, b) then

| | | |
|---|---|---|
| i | Arithmetic mean | $\int_a^b x\, f(x)dx$ |
| ii | Harmonic mean | $\int_a^b \frac{1}{x} f(x)dx$ |
| iii | Geometric mean 'G' log G | $\int_a^b \log x\, f(x)dx$ |
| iv | Moments about origin | $\int_a^b x^r\, f(x)dx$ |
| v | Moments about any point A | $\int_a^b (x-A)^r\, f(x)dx$ |
| vi | Moment about mean $\mu_r$ | $\int_a^b (x-\text{mean})^r\, f(x)dx$ |
| vii | Variance $\mu_2$ | $\int_a^b (x-\text{mean})^2\, f(x)dx$ |
| viii | Mean deviation about the mean is M.D. | $\int_a^b \lvert x-\text{mean}\rvert\, f(x)dx$ |

**3.3 Mathematical Expectations**

**Def :**Let 'X' be a continuous random variable with probability density function f(x). Then the mathematical expectation of 'X' is denoted by E(X) and is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

It is denoted by

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx$$

Thus

$$\mu_1' = E(X) \qquad (\mu_1' \text{ about origin})$$

$$\mu_2' = E(X^2) \qquad (\mu_2' \text{ about origin})$$

$$\therefore \text{Mean} = \overline{X} = \mu_1' = E(X)$$

And

$$\text{Variance} = \mu_2' - \mu_2'^2$$

$$\text{Variance} = E(X^2) - [E(X)]^2 \qquad \text{(a)}$$

* $r^{th}$ moment (abut mean)

Now

$$E\{X - E(X)\}^r = \int_{-\infty}^{\infty} \{x - E(X)\}^r f(x) dx$$

$$= \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x) dx$$

Thus

$$\mu_r = \int_{-\infty}^{\infty} \{x - \overline{X}\}^r f(x) dx \qquad \text{(b)}$$

$$\text{Where} \quad \mu_r = E[X - E(X)^r]$$

This gives the $r^{th}$ moment about mean and it is denoted by $\mu_r$

Put r = 1 in (B) we get

$$\mu_r = \int_{-\infty}^{\infty} \{x - \overline{X}\} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx - \int_{-\infty}^{\infty} \overline{x} f(x)\,dx$$

$$= \overline{X} - \overline{X} \int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= \overline{X} - \overline{X}$$

$$\mu_1 = 0$$

Put r = 2 in (B), we get

$$\mu_2 = \int_{-\infty}^{\infty} (x - \overline{X})^2 f(x)\,dx$$

$$\text{Variance} = \mu_2 = E[X - E(X)]^2$$

Which gives the variance interms of expectations.

**Note**

Let g(x) = K (Constant), then

$$E\left[g(X)\right] = E(K) = \int_{-\infty}^{\infty} K f(x)\,dx$$

$$= K \int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= K \, . \, 1 = K$$

Thus $E(K) = K \Rightarrow E[\text{a constant}] = \text{constant}$.

**3.4 EXPECTATIONS (Discrete R.V.'s)**

Let 'X' be a discrete random variable with P.M.F p(x)

Then

$$E(X) = \sum_x x\, p(x)$$

For discrete random variables 'X'

$$E(X^r) = \sum_x x^r\, p(x) \qquad \text{(by def)}$$

If we denote

$$E(X^r) = \mu_r'$$

Then

$$\mu_r' = E[X^r] = \sum_x x^r p(x)$$

Put r = 1, we get

$$\text{Mean } \mu_r' = \sum x\, p(x)$$

Put r = 2, we get

$$\mu_2' = E[X^2] = \sum_x x^2 p(x)$$

$$\therefore \mu_2 = \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$$

The $r^{th}$ moment about mean

$$\mu_r' = E[\{X - E(X)\}^r]$$

$$= \sum_x (x - \overline{X})^r p(x), \qquad E(X) = \overline{X}$$

Put r = 2, we get

$$\text{Variance} = \mu_2 = \sum_x ((x - \overline{X})^2 p(x)$$

## 3.5 ADDITION THEOREM (EXPECTATION)

**Theorem 1**

If X and Y are two continuous random variable with pdf fx(x) and fy(y) then

E(X+Y) = E(X) + E(Y)

## 3.6 MULTIPLICATION THEOREM OF EXPECTATION

**Theorem 2**

If X and Y are independent random variables, Then E(XY) = E(X) . E(Y)

**Note :**

If X1, X2, ……, Xn are 'n' independent random variables, then

E[X1, X2, ……, Xn] = E(X1), E(X2), ……, E(Xn)

**Theorem 3**

If 'X' is a random variable with pdf f(x) and 'a' is a constant, then

(i) E[a G(x)] = a E[G(x)]

(ii) E[G(x)+a] = E[G(x)+a]

Where G(X) is a function of 'X' which is also a random variable.

**Theorem 4**

If 'X' is a random variable with p.d.f. f(x) and 'a' and 'b' are constants, then E[ax + b] = a E(X) + b

**Cor 1:**

If we take a = 1 and b = –E(X) = – X , then we get

E(X- X ) = E(X) – E(X) = 0

**Note**

$$E\left(\frac{1}{X}\right) \neq \frac{1}{E(X)}$$

$$E[\log (x)] \neq \log E(X)$$

$$E(X^2) \neq [E(X)]^2$$

**3.7 EXPECTATION OF A LINEAR COMBINATION OF RANDOM VARIABLES**

Let $X_1, X_2, \ldots, X_n$ be any 'n' random variable and if $a_1, a_2, \ldots, a_n$ are constants, then
$E[a_1X_1 + a_2X_2 + \ldots + a_nX_n] = a_1E(X_1) + a_2E(X_2) + \ldots + a_nE(X_n)$

**Result**

If X is a random variable, then

$Var(aX + b) = a^2 Var(X)$ 'a' and 'b' are constants.

**Covariance :**

If X and Y are random variables, then covariance between them is defined as $Cov(X, Y) = E\{[X - E(X)][Y - E(Y)]\}$

$Cov(X, Y) = E(XY) - E(X) . E(Y)$ (A)

If X and Y are independent, then

$E(XY) = E(X) E(Y)$

Sub (B) in (A), we get $Cov(X, Y) = 0$

∴ If X and Y are independent, then

$Cov(X, Y) = 0$

Note

(i) $Cov(aX, bY) = ab\, Cov(X, Y)$

(ii) $Cov(X+a, Y+b) = Cov(X, Y)$

(iii) $Cov(aX+b, cY+d) = ac\, Cov(X, Y)$

(iv) $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\, Cov(X_1, X_2)$

If $X_1, X_2$ are independent

$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$

**EXPECTATION TABLE**

| Discrete R.V's | Continuous R.V's |
|---|---|
| 1. $E(X) = \sum x\, p(x)$ | 1. $E(X) = \int_{-\infty}^{\infty} x f(x) dx$ |
| 2. $E(X^r) = \mu_r' = \sum_x x^r p(x)$ | 2. $E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx$ |
| 3. $\text{Mean} = \mu_r' = \sum x\, p(x)$ | 3. $\text{Mean} = \mu_r' = \int_{-\infty}^{\infty} x f(x) dx$ |
| 4. $\mu_2' = \sum x^2 p(x)$ | 4. $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x) dx$ |
| 5. $\text{Variance} = \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$ | 5. $\text{Variance} = \mu_2' - \mu_1'^2 = E(X^2) - \{E($ |

## SOLVED PROBLEMS ON DISCRETE R.V'S

**Example :1**

When die is thrown, 'X' denotes the number that turns up. Find E(X), E($X^2$) and Var (X).

**Solution**

Let 'X' be the R.V. denoting the number that turns up in a die. 'X' takes values 1, 2, 3, 4, 5, 6 and with probability 1/6 for each

| X = x | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| p(x) | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 | 1/6 |
| | $p(x_1)$ | $p(x_2)$ | $p(x_3)$ | $p(x_4)$ | $p(x_5)$ | $p(x_6)$ |

Now

$$\begin{aligned} E(X) &= \sum_{i=1}^{6} x_i\, p(x_i) \\ &= x_1p(x_1) + x_2p(x_2) + x_3p(x_3) + x_4p(x_4) + x_5p(x_5) + x_6p(x_6) \\ &= 1 \text{ x } (1/6) + 1 \text{ x } (1/6) + 3 \text{ x } (1/6) + 4 \text{ x } (1/6) + 5 \text{ x } (1/6) + 6 \text{ x } (1/6) \\ &= 21/6 \quad = \quad 7/2 \qquad (1) \end{aligned}$$

$$\begin{aligned} E(X) &= \sum_{i=1}^{6} x_i\, p(x_p) \\ &= x_1^2p(x_1)+x_2^2p(x_2)+x_3^2p(x_3)+x_4^2p(x_4)+x_5^2p(x_5)+x_6p(x_6) \\ &= 1(1/6) + 4(1/6) + 9(1/6) + 16(1/6) + 25(1/6) + 36(1/6) \\ &= \frac{1+4+9+16+25+36}{6} \quad = \quad \frac{91}{6} \qquad (2) \end{aligned}$$

$$\text{Variance } (X) = \text{Var } (X) = E(X^2) - [E(X)]^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

**Example :2**

Find the value of (i) C (ii) mean of the following distribution given

$$f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

**Solution**

$$\text{Given } f(x)=\begin{cases} C(x-x^2), & 0<x<1 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$\int_{-\infty}^{\infty} f(x)dx=1$$

$$\int_0^1 C(x-x^2)dx=1 \qquad \text{[using (1)] [}\therefore 0<x<1\text{]}$$

$$C\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=1$$

$$C\left[\frac{1}{2}-\frac{1}{3}\right]=1$$

$$C\left[\frac{3-2}{6}\right]=1$$

$$\frac{C}{6}=1 \qquad C=6 \qquad (2)$$

Sub (2) in (1), $f(x) = 6(x - x^2)$, $0< x < 1$ (3)

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\,f(x)dx$$

$$= \int_0^1 x\;6(x-x^2)dx \qquad \text{[from (3)]} \qquad [\therefore 0<x<1]$$

$$= \int_0^1 (6x^2-x^3)dx$$

$$= \left[\frac{6x^3}{3}-\frac{6x^4}{4}\right]_0^1$$

$\therefore$ Mean = ½

| Mean | C |
|---|---|
| ½ | 6 |

# 4 CONTINUOUS DISTRIBUTION FUNCTION

Def :

If f(x) is a p.d.f. of a continuous random variable 'X', then the function

$$F_X(x) = F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)\,dx, \quad -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable.

*** PROPERTIES OF CDF OF A R.V. 'X'**

(i) $0 \le F(x) \le 1, \ -\infty < x < \infty$

(ii) $\underset{x \to -\infty}{Lt} F(x) = 0, \qquad \underset{x \to -\infty}{Lt} F(x) = 1$

(iii) $P(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$

(iv) $F'(x) = \dfrac{dF(x)}{dx} = f(x) \ge 0$

(v) $P(X = x_i) = F(x_i) - F(x_i - 1)$

**Example :1.4.1**

Given the p.d.f. of a continuous random variable 'X' follows

$$f(x)=\begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}, \text{ find c.d.f. for 'X'}$$

**Solution**

$$\text{Given } f(x)=\begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{The c.d.f is } F(x)=\int_{-\infty}^{x} f(x)\,dx,\ -\infty<x<\infty$$

(i) When $x < 0$, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} 0\,dx \quad = 0$$

(ii) When $0< x < 1$, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)\,dx = 6\int_{0}^{x} x(1-x)\,dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$= 3x^2 - 2x^3$$

(iii) When $x > 1$, then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 6x(1-x)\,dx + \int_{0}^{x} 0\,dx$$

$$= 6\int_{0}^{1} (x-x^2)\,dx \quad = 1$$

Using (1), (2) & (3) we get

$$F(x)=\begin{cases} 0, & x<0 \\ 3x^2-2x^3, & 0<x<1 \\ 1, & x>1 \end{cases}$$

**Example:1.4.2**

(i) If $f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ defined as follows a density function ?

(ii) If so determine the probability that the variate having this density will fall in the in
2).

**Solution**

Given $f(x)=\begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) In $(0, \infty)$, $e^{-x}$ is +ve

$\therefore f(x) \geq 0$ in $(0, \infty)$

(b) $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_{0}^{\infty} \qquad = -e^{-\infty} + 1$$

$$= 1$$

Hence f(x) is a p.d.f

(ii) We know that

$$P(a \leq X \leq b) \qquad = \int_{a}^{b} f(x)\,dx$$

$$P(1 \leq X \leq 2) \qquad = \int_{1}^{2} f(x)\,dx \qquad = \int_{1}^{2} e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$$

$$= \int_{1}^{2} e^{-x}\,dx = [-e^{-x}]_{+1}^{2}$$

$$= -e^{-2} + e^{-1} \qquad = -0.135 + 0.368 \qquad = 0.233$$

**Example:1.4..3**

A probability curve y = f(x) has a range from 0 to ∞. If f(x) = $e^{-x}$, find the variance and the third moment about mean.

**Solution**

$$\text{Mean} = \int_0^\infty x\,f(x)\,dx$$

$$= \int_0^\infty x\,e^{-x}\,dx \qquad = \left[x[-e^{-x}]-[e^{-x}]\right]_0^\infty$$

Mean = 1

$$\text{Variance}\ \mu_2 = \int_0^\infty (x-\text{Mean})^2 f(x)\,dx$$

$$= \int_0^\infty (x-1)^2 e^{-x}\,dx$$

$$\mu_2 = 1$$

Third moment about mean

$$\mu_3 = \int_a^b (x-\text{Mean})^3 f(x)\,dx$$

Here a = 0, b = ∞

$$\mu_3 = \int_a^b (x-1)^3 e^{-x}\,dx$$

$$= \left\{(x-1)^3(-e^{-x}) - 3(x-1)^2(e^{-x}) + 6(x-1)(-e^{-x}) - 6(e^{-x})\right\}_0^\infty$$

$$= -1 + 3 - 6 + 6 = 2$$

$$\mu_3 = 2$$

# 5 MOMENT GENERATING FUNCTION

**Def :** The moment generating function (MGF) of a random variable 'X' (about origin) whose probability function f(x) is given by

$M_X(t) \quad = E[e^{tX}]$

$$= \begin{cases} \int\limits_{x=-\infty}^{\infty} e^{tx} f(x)dx, \text{ for a continuous probably function} \\ \sum\limits_{x=-\infty}^{\infty} e^{tx} p(x), \text{ for a discrete probably function} \end{cases}$$

Where t is real parameter and the integration or summation being extended to the entire x.

**Example :1.5.1**

Prove that the $r^{th}$ moment of the R.V. 'X' about origin is $M_X(t) = \int\limits_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}$

**Proof**

$$\text{WKT } M_X(t) = E(e^{tX})$$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots + + \frac{(tX)^r}{r!} + \dots\right]$$

$$= E[1] + tE(X) + \frac{t^2}{2!}E(X^2) + \dots + \frac{t^r}{r!}E(X^r) + \dots\dots$$

$$M_X(t) = 1 + t\mu_1^{'} + \frac{t^2}{2!}\mu_2^{'} + \frac{t^3}{3!}\mu_3^{'} + \dots + \frac{t^r}{r!}\mu_r^{'} + \dots\dots$$

[using $\mu_r^{'} = E(X^r)$]

Thus $r^{th}$ moment = coefficient of $\frac{t^r}{r!}$

**Note**

1. The above results gives MGF interms of moments.
2. Since $M_X(t)$ generates moments, it is known as moment generating function.

**Example:1.5.2**

Find $\mu_1^{'}$ and $\mu_2^{'}$ from $M_X(t)$

**Proof**

WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'}$

$$M_X(t) = \mu_0^{'} + \frac{t}{1!}\mu_1^{'} + \frac{t^2}{2!}\mu_2^{'} + \ldots.. + \frac{t^r}{r!}\mu_r^{'} \qquad (A)$$

Differenting (A) W.R.T 't', we get

$$M_X^{'}(t) = \mu_1^{'} + \frac{2t}{2!}\mu_2^{'} + \frac{t^3}{3!}\mu_3^{'} + \ldots.. \qquad (B)$$

Put $t = 0$ in (B), we get

$$M_X^{'}(0) = \mu_1^{'} = \text{Mean}$$

$$\text{Mean} = M_1^{'}(0) \qquad \text{(or)} \qquad \left[\frac{d}{dt}(M_X(t))\right]_{t=0}$$

$$M_X^{''}(t) = \mu_2^{'} + t\,\mu_3^{'} + \ldots\ldots.$$

Put $t = 0$ in (B)

$$M_X^{''}(0) = \mu_2^{'} \qquad \text{(or)} \qquad \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0}$$

In general $\mu_r^{'} = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}$

**Example :1.5.3**

Obtain the MGF of X about the point X = a.

**Proof**

The moment generating function of X about the point X = a is $M_X(t) \qquad = E[e^{t(X-a}$

$$= E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + .... + \frac{t^r}{r!}(X-a)^r + ....\right]$$

$$\begin{bmatrix} \text{Formula} \\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... \end{bmatrix}$$

$$= E(1) + E[t(X-a)] + E[\frac{t^2}{2!}(X-a)^2] + .... + E[\frac{t^r}{r!}(X-a)^r] + ....$$

$$= 1 + t\,E(X-a) + \frac{t^2}{2!}E(X-a)^2 + .... + \frac{t^r}{r!}E(X-a)^r + ....$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + .... + \frac{t^r}{r!}\mu_r' + .... \qquad \text{Where } \mu_r' = E[(X-a)^r]$$

$$[M_X(t)]_{x=a} \quad = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + ..... + \frac{t^r}{r!}\mu_r' + .....$$

**Result:**

$M_{CX}(t) = E[e^{tcx}]$ (1)

$M_X(t) = E[e^{ctx}]$ (2)

From (1) & (2) we get

$$M_{CX}(t) = M_X(ct)$$

**Example :1.5.4**

If $X_1$, $X_2$, ….., $X_n$ are independent variables, then pro

$$M_{X_1+X_2+....+X_n}(t) = E[e^{t(X_1+X_2+....+X_n)}]$$

$$= E[e^{tX_1}.e^{tX_2}.....e^{tX_n}]$$

$$= E(e^{tX_1}).E(e^{tX_2}).....E(e^{tX_n})$$

[$\therefore$ $X_1$, $X_2$, ….., $X_n$ are independent]

$$= M_{X_1}(t).M_{X_2}(t)..........M_{X_n}(t)$$

**Example:1.5.5**

Prove that if $\cup = \dfrac{X-a}{h}$, then $M_{\cup}(t) = e^{\frac{-at}{h}}.M_X\left(\frac{t}{h}\right)$, where a, h are constants.

**Proof**

By definition

$$M_{\cup}(t) = E\left[e^{tu}\right] \qquad \because \left[M_X(t) = E[e^{tx}]\right]$$

$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$= E\left[e^{\frac{tX}{n}-\frac{ta}{n}}\right]$$

$$= E[e^{\frac{tX}{h}}]\ \ E[e^{\frac{-ta}{h}}]$$

$$= e^{\frac{-ta}{h}}\ E[e^{\frac{tX}{h}}] \qquad \text{[by def]}$$

$$= e^{\frac{-ta}{h}}.\ M_X\left(\frac{t}{h}\right)$$

$\therefore M_{\cup}(t) = e^{\frac{-at}{h}}.M_X\left(\dfrac{t}{h}\right)$, where $\cup = \dfrac{X-a}{h}$ and $M_X(t)$ is the MGF about origin

**Example:1.5.6**

Find the MGF for the distribution where

$$f(x) = \begin{cases} \frac{2}{3} & \text{at } x = 1 \\ \frac{1}{3} & \text{at } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

**Solution**

Given $f(1) = \frac{2}{3}$

$f(2) = \frac{1}{3}$

$f(3) = f(4) = \ldots\ldots = 0$

MGF of a R.V. 'X' is given by

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} f(x)$$

$= e^0 f(0) + e^t f(1) + e^{2t} f(2) + \ldots\ldots$

$= 0 + e^t f(2/3) + e^{2t} f(1/3) + 0$

$= 2/3e^t + 1/3e^{2t}$

$\therefore$ MGF is $M_X(t) = \frac{e^t}{3}[2 + e^t]$

# 6 Discrete Distributions

The important discrete distribution of a random variable 'X' are

1. Binomial Distribution
2. Poisson Distribution
3. Geometric Distribution

## 6.1BINOMIAL DISTRIBUTION

**Def :** A random variable X is said to follow binomial distribution if its probability law is given by

P(x) = p(X = x successes) = nCx px $q^{n-x}$ Where x = 0, 1, 2, ……., n, p+q = 1

**Note**

Assumptions in Binomial distribution

i) There are only two possible outcomes for each trail (success or failure).

ii) The probability of a success is the same for each trail.

iii) There are 'n' trails, where 'n' is a constant.

iv) The 'n' trails are independent.

**Example :1.6.1**

Find the Moment Generating Function (MGF) of a binomial distribution about origin.

**Solution**

WKT $$M_X(t) = \sum_{x=0}^{n} e^{tx} p(x)$$

Let 'X' be a random variable which follows binomial distribution then MGF about given by

$$E[e^{tX}] = M_X(t) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x} \qquad \left[\because p(x) = nC_x p^x q^{n-x}\right]$$

$$= \sum_{x=0}^{n} (e^{tx}) p^x nC_x q^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x nC_x q^{n-x}$$

$$\therefore M_X(t) = (q + pe^t)^n$$

**Example:1.6.2**
Find the mean and variance of binomial distribution.
**Solution**

$$M_X(t) = (q+pe^t)^n$$
$$\therefore M'_X(t) = n(q+pe^t)^{n-1}.pe^t$$

Put $t = 0$, we get

$$M'_X(0) = n(q+p)^{n-1}.p$$

$$\text{Mean} = E(X) = np \qquad [\because (q+p)=1] \qquad [\text{Mean } M'_X(0)]$$

$$M''_X(t) = np\left[(q+pe^t)^{n-1}.e^t + e^t(n-1)(q+pe^t)^{n-2}.pe^t\right]$$

Put $t = 0$, we get

$$M''_X(t) = np\left[(q+p)^{n-1} + (n-1)(q+p)^{n-2}.p\right]$$
$$= np[1+(n-1)p]$$
$$= np + n^2p^2 - np^2$$
$$= n^2p^2 + np(1-p)$$
$$M''_X(0) = n^2p^2 + npq \qquad [\because 1-p=q]$$
$$M''_X(0) = E(X^2) = n^2p^2 + npq$$
$$\text{Var}(X) = E(X^2) - [E(X)]^2 = n^2/p^2 + npq - n^2/p^2 = npq$$
$$\text{Var}(X) = npq$$
$$\text{S.D} = \sqrt{npq}$$

**Example :1.6.3**

Find the Moment Generating Function (MGF) of a binomial distribution abo
(np).

**Solution**

Wkt the MGF of a random variable X about any point 'a' is

$M_x(t)$ (about X = a) $= E[e^{t(X-a)}]$

Here 'a' is mean of the binomial distribution

$M_X(t)$ (about X = np) $= E[e^{t(X-np)}]$

$$= E[e^{tX} . e^{-tnp)}]$$
$$= e^{-tnp} . [-[e^{-tX)}]]$$
$$= e^{-tnp} . (q+pe^t)-^n$$
$$= (e^{-tp})^n . (q+pe^t)^n$$

$\therefore$ MGF about mean $= (e^{-tp})^n . (q+pe^t)^n$

**Example :1.6.4**

Additive property of binomial distribution.

**Solution**

The sum of two binomial variants is not a binomial variate.

Let X and Y be two independent binomial variates with p
$(n_1, p_1)$ and $(n_2, p_2)$ respectively.

Then

$$M_X(t) = \left(q_1 + p_1e^t\right)^{n_1}, \qquad M_Y(t) = \left(q_2 + p_2e^t\right)^{n_2}$$

$$\therefore\ M_{X+Y}(t) = M_X(t).M_Y(t) \quad [\because \text{X \& Y are independent R.V.'s}]$$

$$= \left(q_1 + p_1e^t\right)^{n_1}.\left(q_2 + p_2e^t\right)^{n_2}$$

RHS cannot be expressed in the form $\left(q + pe^t\right)^n$. Hence by uniqueness th
MGF X+Y is not a binomial variate. Hence in general, the sum of two binomial variate
binomial variate.

**Example :1.6.5**

If $M_X(t) = \left(q+pe^t\right)^{n_1}$, $M_Y(t) = \left(q+pe^t\right)^{n_2}$, then

$$M_{X+Y}(t) = \left(q+pe^t\right)^{n_1+n_2}$$

**Problems on Binomial Distribution**

1. Check whether the following data follow a binomial distribution or not. Mean = 3; 
4.

**Solution**

Given Mean $np = 3$ (1)

Variance $npr = 4$ (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{3}{4}$$

$$\Rightarrow q = \frac{4}{3} = 1\frac{1}{3} \text{ which is} > 1.$$

Since $q > 1$ which is not possible $(0 < q < 1)$. The given data not follow binomial distri

**Example :1.6.5**

The mean and SD of a binomial distribution are 5 and 2, determine the distributi

**Solution**

Given Mean = np = 5 (1)

$SD = \sqrt{npq} = 2$ (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$\therefore p = 1 - \frac{4}{5} = \frac{1}{5} \Rightarrow p = \frac{1}{5}$$

Sub (3) in (1) we get

n x 1/5 = 5

n = 25

$\therefore$ The binomial distribution is

$$P(X = x) = p(x) = nC_x p^x q^{n-x}$$
$$= 25C_x(1/5)^x(4/5)^{n-x}, \quad x = 0, 1, 2, \ldots, 25$$

# 7 Passion Distribution

**Def :**

A random variable X is said to follow if its probability law is given by

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

Poisson distribution is a limiting case of binomial distribution under the following conditions or assumptions.

1. The number of trails 'n' should e infinitely large i.e. $n \to \infty$.
2. The probability of successes 'p' for each trail is infinitely small.
3. $np = \lambda$ , should be finite where $\lambda$ is a constant.

*** To find MGF**

*** To find MGF**

$$M_X(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} \left( \frac{\lambda^x e^{\lambda}}{x!} \right)$$
$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} (\lambda e^t)^x}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$
$$= e^{-\lambda} \left[ 1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \dots\dots \right]$$
$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Hence $M_X(t) = e^{\lambda(e^t - 1)}$

* To find Mean and Variance

WKT $M_X(t) = e^{\lambda(e^t - 1)}$

$\therefore$ $M_X'(t) = e^{\lambda(e^t - 1)} . e^t$

$M_X'(0) = e^{-\lambda} . \lambda$

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x . p(x)$$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^{x}}{x!} \quad = \sum_{x=0}^{\infty} \frac{x.\,e^{-\lambda}\lambda\lambda^{x-1}}{x!}$$

$$= 0 + e^{-\lambda}.\lambda\sum_{x=1}^{\infty} \frac{x.\lambda^{x-1}}{x!}$$

$$= \lambda e^{-\lambda}.\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left[1+\lambda+\frac{\lambda^{2}}{2!}+.....\right]$$

$$= \lambda e^{-\lambda}.e^{\lambda}$$

Mean $= \lambda$

$$\mu_2' \quad = E[X^2] \quad = \sum_{x=0}^{\infty} x^2.p(x) \quad = \sum_{x=0}^{\infty} x^2.\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\}.\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} \frac{x.e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\lambda^{2}\sum_{x=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)(x-3)....1} + \lambda$$

$$= e^{-\lambda}\lambda^{2}\sum_{x=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^{2}\left[1+\frac{\lambda}{1!}+\frac{\lambda^{2}}{2!}+....\right]+\lambda$$

$$= \lambda^2 + \lambda$$

Variance $\mu_2 = E(X^2) - [E(X)]^2 \quad = \lambda^2 + \lambda - \lambda^2 = \lambda$

Variance $= \lambda$

Hence Mean $=$ Variance $= \lambda$

Note : * sum of independent Poisson Vairates is also Poisson variate.

**PROBLEMS ON POISSON DISTRIBUTION**

## PROBLEMS ON POISSON DISTRIBUTION

**Example:1.7.1**

If x is a Poisson variate such that $P(X=1) = \frac{3}{10}$ and $P(X=2) = \frac{1}{5}$, find the P(X=0) and

**Solution**

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\therefore P(X=1) = e^{-\lambda}\lambda = \frac{3}{10} \qquad \text{(Given)}$$

$$= \lambda e^{-\lambda} = \frac{3}{10} \qquad (1)$$

$$P(X=2) = \frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{1}{5} \qquad \text{(Given)}$$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = \frac{1}{5} \qquad (2)$$

$$(1) \Rightarrow e^{-\lambda}\lambda = \frac{3}{10} \qquad (3)$$

$$(2) \Rightarrow e^{-\lambda}\lambda^{2} = \frac{2}{5} \qquad (4)$$

$$\frac{(3)}{(4)} \Rightarrow \frac{1}{\lambda} = \frac{3}{4}$$

$$\lambda = \frac{4}{3}$$

$$\therefore \quad P(X=0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-4/3}$$

$$P(X=3) = \frac{e^{-\lambda}\lambda^{3}}{3!} = \frac{e^{-4/3}(4/3)^{3}}{3!}$$

**Example :1.7.2**

If X is a Poisson variable

$P(X = 2) \quad = 9\,P(X = 4) + 90\,P(X=6)$

Find (i) Mean if X (ii) Variance of X

**Solution**

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0,1,2,\ldots..$$

Given $P(X = 2) = 9\,P(X = 4) + 90\,P(X=6)$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = 9\frac{e^{-\lambda}\lambda^{4}}{4!} + 90\frac{e^{-\lambda}\lambda^{6}}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^{2}}{4!} + \frac{90\lambda^{4}}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^{2}}{8} + \frac{\lambda^{4}}{8}$$

$$1 = \frac{3\lambda^{2}}{4} + \frac{\lambda^{4}}{4}$$

$$\lambda^{4} + 3\lambda^{2} - 4 = 0$$

$\lambda^{2} = 1$ or $\lambda^{2} = -4$

$\lambda = \pm 1$ or $\lambda = \pm 2i$

$\therefore$ Mean $= \lambda = 1$, Variance $= \lambda = 1$

$\therefore$ Standard Deviation = 1

**7.3 Derive probability mass function of Poisson distribution as a limiting case of Binomial distribution**

**Solution**

We know that the Binomial distribution is $P(X=x) = nCx\ p^{x}q^{n-x}$

$$P(X=x) = nC_x\, p^x q^{n-x}$$
$$= \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$
$$= \frac{1.2.3.......(n-x)(n-x+1)......np^n}{1.2.3.....(n-x)\;\; x!} \frac{(1-p)^n}{(1-p)^x}$$
$$= \frac{1.2.3.......(n-x)(n-x+1)......n}{1.2.3.....(n-x)\;\; x!} \left(\frac{p}{1-p}\right)^x (1-p)^n$$
$$= \frac{n(n-1)(n-2)......(n-x+1)}{x!}\; \frac{\lambda^x}{n^x} \frac{1}{\left(1-\frac{\lambda}{n}\right)^x} \left(1-\frac{\lambda}{n}\right)^n$$
$$= \frac{n(n-1)(n-2)......(n-x+1)}{x!}\; x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$
$$P(X=x) = \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}}{x!} \lambda^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$
$$= \frac{\lambda^x}{x!} 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

When $n \to \infty$

$$P(X=x) = \frac{\lambda^x}{x!} \operatorname*{lt}_{n\to\infty}\left[1-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)......\left\{1-\left(\frac{x-1}{n}\right)\right\}\left(1-\frac{\lambda}{n}\right)^{n-x}\right]$$
$$= \frac{\lambda^x}{x!} \operatorname*{lt}_{n\to\infty}\left(1-\frac{1}{n}\right)\operatorname*{lt}_{n\to\infty}\left(1-\frac{2}{n}\right)......\operatorname*{lt}_{n\to\infty} 1-\left(\frac{x-1}{n}\right)$$

We know that

$$\operatorname*{lt}_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$$

and $$\operatorname*{lt}_{n\to\infty}\left(1-\frac{1}{n}\right) = \operatorname*{lt}_{n\to\infty}\left(1-\frac{2}{n}\right)..... = \operatorname*{lt}_{n\to\infty}\left(1-\left(\frac{x-1}{n}\right)\right) = 1$$

$$\therefore P(X=x) = \frac{\lambda^x}{x!} e^{-\lambda}, x = 0,1,2,......\infty$$

# 8 GEOMETRIC DISTRIBUTION

Def: A discrete random variable 'X' is said to follow geometric distribution, if it assumes only non-negative values and its probability mass function is given by

$P(X=x) = p(x) = q^{x-1}$ ; $x = 1, 2, \ldots\ldots, 0 < p < 1,$ Where $q = 1-p$

**Example:1.8.1**

To find MGF

$$M_X(t) = E[e^{tx}]$$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x} q^{-1} p$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x} p / q$$

$$= p/q \sum_{x=1}^{\infty} e^{tx} q^{x}$$

$$= p/q \sum_{x=1}^{\infty} (e^{t}q)^{x}$$

$$= p/q \left[ (e^{t}q)^{1} + (e^{t}q)^{2} + (e^{t}q)^{3} + \ldots \right]$$

Let $x = e^{t}q$ $= p/q \left[ x + x^{2} + x^{3} + \ldots \right]$

$$= \frac{p}{q} x \left[ 1 + x + x^{2} + \ldots \right] \quad = \frac{p}{q} (1-x)^{-1}$$

$$= \frac{p}{q} qe^{t} \left[ 1 - qe^{t} \right] \quad = pe^{t} \, [1 - qe^{t}]^{-1}$$

$$\therefore M_X(t) = \frac{pe^{t}}{1-qe^{t}}$$

*** To find the Mean & Variance**

$$M_X'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$\therefore E(X) = M_X'(0) = 1/p$$

$$\therefore \text{Mean} = 1/p$$

$$\text{Variance} \qquad \mu_X''(t) = \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]$$

$$= \frac{(1-qe^t)^2 pe^t - pe^t 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)^2 pe^t + 2pe^t qe^t(1-qe^t)}{(1-qe^t)^4}$$

$$M_X''(0) = \frac{1+q}{p^2}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{(1+q)}{p^2} - \frac{1}{p^2} \Rightarrow \frac{q}{p^2}$$

$$\text{Var}(X) = \frac{q}{p^2}$$

**Note:**

Another form of geometric distribution

$P[X=x] = q^x p$ ; $x = 0, 1, 2, \ldots$

$$M_X(t) = \frac{p}{(1-qe^t)}$$

Mean $= q/p$, Variance $= q/p^2$

**Example:1.8.2**

If the MGF of X is $(5\text{-}4et)^{-1}$, find the distribution of X and P(X=5)

**Solution**

Let the geometric distribution be

$P(X = x) = q^x p, \qquad x = 0, 1, 2, \ldots$

The MGF of geometric distribution is given by

$$\frac{p}{1-qe^t} \qquad (1)$$

$$\text{Here } M_X(t) = (5 - 4e^t)^{-1} \Rightarrow 5^{-1}\left[1-\frac{4}{5}e^t\right]^{1} \qquad (2)$$

Company (1) & (2) we get $\quad q=\frac{4}{5};\ p=\frac{1}{5}$

$\therefore P(X = x) \quad = pq^x, \qquad x = 0, 1, 2, 3, \ldots$

$$= \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^x$$

$$P(X = 5) \qquad = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5 = \frac{4^5}{5^6}$$

# 9 CONTINUOUS DISTRIBUTIONS

If ‘X’ is a continuous random variable then we have the following distribution

1. Uniform (Rectangular Distribution)
2. Exponential Distribution
3. Gamma Distribution
4. Normal Distribution

### 9.1 Uniform Distribution (Rectangular Distribution)

**Def :** A random variable X is set to follow uniform distribution if its

**Def :** A random variable X is set to follow uniform distribution if its

$$f(x)=\begin{cases} \dfrac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

*** To find MGF**

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_b^a$$

$$= \frac{1}{(b-a)t}\left[e^{bx} - e^{at}\right]$$

$\therefore$ The MGF of uniform distribution is

$$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

* To find Mean and Variance

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$= \int_a^b b_x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x\,dx = \frac{\left(\frac{x^2}{2}\right)_a^b}{b-a}$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} = \frac{a+b}{2}$$

$$\text{Mean} \quad \mu_1' = \frac{a+b}{2}$$

Putting $r = 2$ in (A), we get

$$\mu_2' = \int_a^b x^2 \, f(x)dx = \int_a^b \frac{x^2}{b-a} dx$$

$$= \frac{a^2 + ab + b^2}{3}$$

$$\therefore \text{Variance} = \mu_2' - \mu_1'^2$$

$$= \frac{b^2 + ab + b^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12}$$

$$\text{Variance} = \frac{(b-a)^2}{12}$$

**PROBLEMS ON UNIFORM DISTRIBUTION**

**Example 1.9.1**

If X is uniformly distributed over $(-\alpha,\alpha)$, $\alpha < 0$, find $\alpha$ so that

(i) $P(X>1) = 1/3$

(ii) $P(|X| < 1) = P(|X| > 1)$

**Solution**

If X is uniformly distributed in $(-\alpha, \alpha)$, then its p.d.f. is

$$f(x) = \begin{cases} \frac{1}{2\alpha} & -\alpha < x < \alpha \\ 0 & \text{otherwise} \end{cases}$$

(i) P(X>1) = 1/3

$$\int_1^\alpha f(x)dx = 1/3$$

$$\int_1^\alpha \frac{1}{2\alpha} dx = 1/3$$

$$\frac{1}{2\alpha}(x)_1^\alpha = 1/3 \qquad \Rightarrow \frac{1}{2\alpha}(\alpha - 1) = 1/3$$

$$\alpha = 3$$

(ii) P(|X| < 1) = P(|X| > 1) = 1 - P(|X| < 1)

P(|X| < 1) + P(|X| < 1) = 1

2 P(|X| < 1) = 1

2 P(-1 < X < 1) = 1

$$2\int_{-1}^{1} f(x)dx = 1$$

$$2\int_{-1}^{1} \frac{1}{2\alpha} dx = 1$$

$$\Rightarrow \alpha = 2$$

**Note:**

1. The distribution function F(x) is given by

$$F(x) = \begin{cases} 0 & -\alpha < x < \alpha \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & b < x < \infty \end{cases}$$

2. The p.d.f. of a uniform variate 'X' in (-a, a) is given by

$$F(x) = \begin{cases} \frac{1}{2a} & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

# 10 THE EXPONENTIAL DISTRIBUTION

**Def :**A continuous random variable 'X' is said to follow an exponential distribution with parameter λ>0 if its probability density function is given by

$$F(x) = \begin{cases} \lambda e^{-\lambda x} & x > a \\ 0 & \text{otherwise} \end{cases}$$

**To find MGF**

**Solution**

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}dx \qquad = \lambda \int_0^{\infty} e^{-(\lambda-t)x}dx$$

$$= \lambda\left[\frac{e^{-(\lambda-t)x}}{\lambda-t}\right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda-t)}\left[e^{-\infty} - e^{-0}\right] \qquad = \frac{\lambda}{\lambda-t}$$

$$\therefore \text{ MGF of } x = \frac{\lambda}{\lambda-t}, \lambda > t$$

*** To find Mean and Variance**

We know that MGF is

$$M_X(t) = \frac{\lambda}{\lambda - t} = \frac{1}{1-\frac{t}{\lambda}} = \left(1-\frac{t}{\lambda}\right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \ldots\ldots + \frac{t^r}{\lambda^r}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{2!}\left(\frac{2!}{\lambda^2}\right) + \ldots\ldots + \frac{t^r}{r!}\left(\frac{t!}{\lambda^r}\right)$$

$$M_X(t) = \sum_{r=0}^{\infty}\left(\frac{t}{\lambda}\right)^r$$

$\therefore$ Mean $\quad \mu_1' = \text{Coefficient of } \frac{t^1}{1!} = \frac{1}{\lambda}$

$\mu_2' = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{\lambda^2}$

Variance= $\mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Variance = $\frac{1}{\lambda^2}$ $\qquad$ Mean = $\frac{1}{\lambda}$

**Example: 1.10.1**

Let 'X' be a random variable with p.d.f

$$F(x) = \begin{cases} \frac{1}{3}e^{\frac{-x}{3}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $\qquad$ 1) $P(X > 3)$ $\qquad$ 2) MGF of 'X'

**Solution**

WKT the exponential distribution is

$F(x) = \lambda e^{-\lambda x}, \quad x > 0$

Here $\lambda = \frac{1}{3}$

$P(x>3) = \int_3^{\infty} f(x)dx \qquad = \int_3^{\infty} \frac{1}{3}e^{-\frac{x}{3}}dx$

$P(X>3) = e^{-1}$

MGF is $\qquad M_X(t) \qquad = \frac{\lambda}{\lambda - t}$

$$= \frac{\frac{1}{3}}{\frac{1}{3} - t} \qquad = \frac{\frac{1}{3}}{\frac{1-3t}{3}} \qquad = \frac{1}{1-3t}$$

$$M_X(t) = \frac{1}{1-3t}$$

**Note**

If X is exponentially distributed, then

$P(X > s+t \,/\, x > s) \quad = P(X > t)$, for any $s, t > 0$.

# 11 GAMMA DISTRIBUTION

**Definition**

A Continuous random variable X taking non-negative values is said to follow gamma distribution , if its probability density function is given by

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\, dx, \quad k \in (0, \infty)$$

$f(x) = \underline{\qquad\quad}$, $\alpha > 0$, $0 < x < \infty$

$= 0$, elsewhere

When α is the parameter of the distribution.

**Additive property of Gamma Variates**

If X1,X2 , X3,.... Xk are independent gamma variates with parameters λ1,λ2,…..λ krespectively then X1+X2 + X3+.... +Xk is also a gamma variates with parameter λ1+ λ2 +….. + λk

**Example :1.11.1**

Customer demand for milk in a certain locality ,per month , is Known to be a general Gamma RV.If the average demand is a liters and the most likely demand b liters (b<a) , what is the varience of the demand?

Solution :

Let X be represent the monthly Customer demand for milk. Average demand is the value of E(X).

Most likely demand is the value of the mode of X or the value of X for which its density function is maximum.

If f(x) is the its density function of X ,then

$$f(x) = —— .\ x^{k-1}\ e^{-\lambda x}\ \ e^{-\lambda x}\ \ , x>0$$

$$f(x) = —— .[(k-1)\ x^{k-2}\ e^{-\lambda x} - \qquad e^{-\lambda x}\ ]$$

$$= 0 \text{ ,when } x=0 \text{ , } x=——$$

$$f'(x) = —— .[(k-1)\ x^{k-2}\ e^{-\lambda x} - \qquad e^{-\lambda x}\ ]$$

$<0$ , when $x=$——

Therefour f(x) is maximum , when $x=$——

i.e ,Most likely demand =——= b ….(1)

and E(X) = – ………(2)

Now V(X) = = —= – –

= a (a-b) From (1) and (2)

**TUTORIAL QUESTIONS**

1.It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, fine the no. of packets containing at least, exactly and atmost 2 defective items in a consignment of 1000 packets using (i) Binomial distribution (ii) Poisson approximation to binomial distribution.

2. The daily consumption of milk in excess of 20,000 gallons is approximately exponentially distributed with . $3000 = \theta$ The city has a daily stock of 35,000 gallons. What is the probability that of two days selected at random, the stock is insufficient for both days.

3.The density function of a random variable X is given by $f(x)= KX(2-X), 0\leq X\leq 2$.Find K, mean, variance and rth moment.

4.A binomial variable X satisfies the relation $9P(X=4)=P(X=2)$ when $n=6$. Find the parameter p of the Binomial distribution.

5. Find the M.G.F for Poisson Distribution.

6. If X and Y are independent Poisson variates such that $P(X=1)=P(X=2)$ and $P(Y=2)=P(Y=3)$. Find $V(X-2Y)$.

7.A discrete random variable has the following probability distribution

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| X: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| P(X) | a | 3a | 5a | 7a | 9a | 11a | 13a | 15a | 17a |

Find the value of a, $P(X<3)$ and c.d.f of X.

7. In a component manufacturing industry, there is a small probability of 1/500 for any component to be defective. The components are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (1). No defective. (2). Two defective components in a consignment of 10,000 packets.

## WORKED OUT EXAMPLES

**Example 1**

Given the p.d.f. of a continuous random variable 'X' follows

$$f(x)=\begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}, \text{ find c.d.f. for 'X'}$$

**Solution**

Given $f(x)=\begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

The c.d.f is $F(x)=\int_{-\infty}^{x} f(x)dx, -\infty<x<\infty$

(i) When $x<0$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{x} 0\, dx \qquad = 0$$

(ii) When $0<x<1$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)dx = 6\int_{0}^{x} x(1-x)dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$= 3x^2-2x^3$$

(iii) When $x>1$, then

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 6x(1-x)dx + \int_{0}^{x} 0\, dx$$

$$= 6\int_{0}^{1}(x-x^2)dx \qquad = 1$$

Using (1), (2) & (3) we get

$$F(x)=\begin{cases} 0, & x<0 \\ 3x^2-2x^3, & 0<x<1 \\ 1, & x>1 \end{cases}$$

**Example :2**

A random variable X has the following probability function

| Values of X | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Probability P(X) | | a | a | a | a | 1a | 3a | 5a | 7a |

(i) Determine the value of 'a'
(ii) Find $P(X<3)$, $P(X\geq 3)$, $P(0<X<5)$
(iii) Find the distribution function of X.

**Solution**

Table 1

| Values of X | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| p(x) | | a | a | a | a | 1a | 3a | 5a | 7a |

(i) We know that if p(x) is the probability of mass function then

$$\sum_{i=0}^{8} p(x_i) = 1$$

$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) + p(6) + p(7) + p(8) = 1$
$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$
$81\,a = 1$
$a = 1/81$

put a = 1/81 in table 1, e get table 2

**Table 2**

| X = x | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| P (x) | /81 | /81 | /81 | /81 | /81 | 1/81 | 3/81 | 5/81 | 7/81 |

(ii) $P(X < 3) = p(0) + p(1) + p(2)$
$= 1/81 + 3/81 + 5/81 = 9/81$
(ii) $P(X \geq 3) = 1 - p(X < 3)$
$= 1 - 9/81 = 72/81$
(iii) $P(0 < x < 5) = p(1) + p(2) + p(3) + p(4)$ here 0 & 5 are not include
$= 3/81 + 5/81 + 7/81 + 9/81$

$$= \frac{3 + 5 + 7 + 8 + 9}{81} = \frac{24}{81}$$

(iv) To find the distribution function of X using table 2, we get

| **X = x** | **F(X) = P(x ≤ x)** |
|---|---|
| 0 | $F(0) = p(0) = 1/81$ |
| 1 | $F(1) = P(X \leq 1) = p(0) + p(1)$<br>$= 1/81 + 3/81 = 4/81$ |
| 2 | $F(2) = P(X \leq 2) = p(0) + p(1) + p(2)$<br>$= 4/81 + 5/81 = 9/81$ |
| 3 | $F(3) = P(X \leq 3) = p(0) + p(1) + p(2) + p(3)$<br>$= 9/81 + 7/81 = 16/81$ |
| 4 | $F(4) = P(X \leq 4) = p(0) + p(1) + \ldots + p(4)$<br>$= 16/81 + 9/81 = 25/81$ |
| 5 | $F(5) = P(X \leq 5) = p(0) + p(1) + \ldots + p(4) + p(5)$<br>$= 2/81 + 11/81 = 36/81$ |
| 6 | $F(6) = P(X \leq 6) = p(0) + p(1) + \ldots + p(6)$<br>$= 36/81 + 13/81 = 49/81$ |
| 7 | $F(7) = P(X \leq 7) = p(0) + p(1) + \ldots + p(6) + p(7)$<br>$= 49/81 + 15/81 = 64/81$ |
| 8 | $F(8) = P(X \leq 8) = p(0) + p(1) + \ldots + p(6) + p(7) + p(8)$<br>$= 64/81 + 17/81 = 81/81 = 1$ |

**Example :3**

The mean and SD of a binomial distribution are 5 and 2, determine the distribution.

**Solution**

Given Mean = np = 5 (1)

$SD = \sqrt{npq} = 2$ (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{np}{npq} = \frac{4}{5} \Rightarrow q = \frac{4}{5}$$

$$\therefore p = 1 - \frac{4}{5} = \frac{1}{5} \quad \Rightarrow \quad p = \frac{1}{5}$$

Sub (3) in (1) we get

n x 1/5 = 5

n = 25

$\therefore$ The binomial distribution is

$P(X = x) = p(x) = nC_x p^x q^{n-x}$

$= 25C_x(1/5)^x(4/5)^{n-x}, \quad x = 0, 1, 2, \ldots, 25$

**Example :4**

If X is a Poisson variable

$P(X = 2) = 9\, P(X = 4) + 90\, P(X=6)$

Find (i) Mean if X (ii) Variance of X

**Solution**

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x = 0,1,2,\ldots..$$

Given $P(X = 2) = 9\, P(X = 4) + 90\, P(X=6)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$\lambda^2 = 1$ or $\lambda^2 = -4$

$\lambda = \pm 1$ or $\lambda = \pm 2i$

$\therefore$ Mean = $\lambda$ = 1, Variance = $\lambda$ = 1

$\therefore$ Standard Deviation = 1

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