Unit - 5

Linear Systems with Random inputs

Linear System:

A System with functional relationship $f\{x(t)\}$ is linear, if, for any two inputs $x_1(t)$ and $x_2(t)$, the output of the System can be defined as $f\{(a_1 x_1(t) + a_2 x_2(t)\} = a_1 f\{x_1(t)\} + a_2 f\{x_2(t)\}$ where a_1 and a_2 are constants

Time invariance:

Time invariance is defined as a property of linear Systems that if the input is time shifted by an amount τ, the corresponding output will also be time shifted by the same amount

(i.e) if $f\{x(t)\} = y(t)$ then

$$f\{x(t-\tau)\} = y(t-\tau), \quad -\infty < \tau < \infty$$

A system that does not meet the condition is called time Varying System.

Linear time invariant system:

Property : 1

Show that $S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$ Where $S_{xx}(\omega)$ and $S_{yy}(\omega)$ are the power spectral density functions of the input $x(t)$ and the output $y(t)$ and $H(\omega)$ is the System transfer function

Proof :

$$y(t) = \int^{\infty} h(u)\, x(t-u)\, du$$

Property: 2

If the input $x(t)$ and its output $y(t)$ are related by $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then the system is linear time invariant system

Proof

First, we prove the linearity,

consider, $x(t) = a_1 x_1(t) + a_2 x_2(t)$ ——— (1)

then $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du.$

$$= \int_{-\infty}^{\infty} h(u) \left[a_1 x_1(t-u) + a_2 x_2(t-u) \right] du$$

$$= a_1 \int_{-\infty}^{\infty} h(u) x_1(t-u)\, du + a_2 \int_{-\infty}^{\infty} h(u) x_2(t-u)\, du$$

$$= a_1\, y_1(t) + a_2\, y_2(t)$$

∴ The System is linear.

Now, we prove that the System is a time invariant System

Replace t by $t+k$

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x[(t+k)-u]\, du$$

$$= y(t+k)$$

The System is time invariant.

Hence the System is linear time invariant System.

Property : 3

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then $R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$

Proof:

Gn $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ —— (1)

WKT

$$R_{xy}(\tau) = E[\, x(t)\, y(t+\tau)\,]$$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du \right]$$

[by (1)]

$$= \int_{-\infty}^{\infty} E[\, x(t)\, x(t+\tau-u)\,]\, h(u)\, du$$

$t-t-\tau+u$
$-(\tau-u)$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-u)\, h(u)\, du$$

[$\because$ $x(t)$ is WSS]

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$ [by convolution]

Property : 4

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$, then $R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau)$ where $*$ denotes the convolution.

Proof :

Gn $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ —— (1)

$$R_{yy}(\tau) = E[\, y(t)\, y(t+\tau)\,]$$

$$= E\left[\int_{-\infty}^{\infty} h(u)\, x(t-u)\, y(t+\tau)\, du \right]$$

$$= \int_{-\infty}^{\infty} E[x(t-u)\, y(t+\tau)\, h(u)\, du]$$

$\cancel{t}-u-t-\tau$
$-(\tau+u)$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau+u)\, h(u)\, du$$

Put $u = -\alpha$

$du = -d\alpha$

$$= \int_{\infty}^{-\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, (-d\alpha)$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau-\alpha)\, h(-\alpha)\, d\alpha$$

$$= R_{xy}(\tau) * h(-\tau)$$

Property 5

If $\{x(t)\}$ is a WSS process and if $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ then $S_{xy}(\omega) = S_{xx}(\omega) * H(\omega)$

Proof:

$$y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$$

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= E\left[x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du\right]$$

$$= \int_{-\infty}^{\infty} E[x(t)\, x(t+\tau-u)\, h(u)\, du$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau-u)\, h(u)\, du$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

Taking Fourier Transform

$$= F[R_{xx}(\tau)]\, F[h(\tau)]$$

$$S_{xy}(\omega) = S_{xx}(\omega)\, H(\omega) \quad [\text{by defn of Spectrum}]$$

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(1) Show that $\{x(t)\}$ is a WSS process then the output $\{y(t)\}$ is a WSS process.

Soln:

If the input to a time invariant, stable linear system is a WSS process, then the output will also be a WSS process

(i.e) To show that if $\{x(t)\}$ is a WSS process then the output $\{y(t)\}$ is a WSS process.

WKT the input and output are related by $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$ —— (1)

$$E[y(t)] = \int_{-\infty}^{\infty} h(u)\, E[x(t-u)]\, du$$

$\because \{x(t)\}$ is a WSS process, Mean is constant

(i.e) $E[x(t-u)] = \text{constant}$

Hence $E[y(t)] = E[x'(t-u)] \int_{-\infty}^{\infty} h(u)\, du$

$$= \overline{X_u} \int_{-\infty}^{\infty} h(u)\, du$$

= a finite constant, independent of t

[$\because$ System is Stable]

$E[y(t)] = \text{contant}$

Next to ST $R_{yy}(t, t+\tau)$ depends only on τ.

$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, X(t-u_1)\, h(u_2)\, X(t+\tau-u_2)\, \underbrace{du_1\, du_2}_{②}\right]$

[by ①]

$\therefore \{X(t)\}$ is a WSS process.

$E[X(t-u_1)\, X(t+\tau-u_2)]$ is a function of τ,

Say $g(\tau)$.

$② \Rightarrow R_{YY}(t, t+\tau) = g(\tau)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u_1)\, h(u_2)\, du_1\, du_2$

$= $ a function of τ

$\therefore$ The output $\{Y(t)\}$ is also WSS process.

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② Find the Mean Square Value of the processes whose power Spectral density is as given below. $\dfrac{1}{\omega^4+10\omega^2+9}$. To find the Mean Square Value of the process, we can find its auto Correlation function and Substitute $\tau = 0$

Soln

$$S_X(\omega) = \frac{1}{\omega^4+10\omega^2+9}$$

$$= \frac{1}{(\omega^2+9)(\omega^2+1)} = \frac{A}{\omega^2+9} + \frac{B}{\omega^2+1}$$

$$\frac{A}{\omega^2+9} + \frac{B}{\omega^2+1} = \frac{A(\omega^2+1) + B(\omega^2+9)}{(\omega^2+9)(\omega^2+1)}$$

$$\frac{1}{(\omega^2+9)\cancel{(\omega^2+1)}} = \frac{A(\omega^2+1) + B(\omega^2+9)}{(\omega^2+9)\cancel{(\omega^2+1)}}$$

$$1 = A(\omega^2+1) + B(\omega^2+9)$$

Put $\omega^2 = -9$ | Put $\omega^2 = -1$

$1 = 0 + B(8)$

$$\frac{1}{(\omega^2+9)(\omega^2+1)} = \frac{-1/8}{\omega^2+9} + \frac{1/8}{\omega^2+1}$$

$$= \frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right]$$

$R_{xx}(\tau)$ is fourier inverse transform of

$$\frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right] \qquad \left[\because F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}\right]$$

$$R_{xx}(\tau) = F^{-1}\left\{\frac{1}{8}\left[\frac{1}{\omega^2+1} - \frac{1}{\omega^2+9}\right]\right\}$$

$$= \frac{1}{8}F^{-1}\left[\frac{1}{\omega^2+1}\right] - \frac{1}{8}F^{-1}\left[\frac{1}{\omega^2+3^2}\right]$$

$$= \frac{1}{8}\cdot\frac{1}{2}e^{-|\tau|} - \frac{1}{8}\cdot\frac{1}{6}e^{-3|\tau|}$$

$$= \frac{1}{16}e^{-|\tau|} - \frac{1}{48}e^{-3|\tau|}$$

The Mean square value is $R_{xx}(\tau)$ at $\tau = 0$.

$$R_{xx}(0) = \frac{1}{16} - \frac{1}{48} = \frac{2}{48} = \frac{1}{24}$$

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(3) A system has an impulse response $h(t) = e^{-\beta t}U(t)$, find the power spectral density of the output $Y(t)$ corresponding to the input $X(t)$.

soln

Gn $X(t) \longrightarrow$ input process

$Y(t) \longrightarrow$ output process

WKT $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$ —— (1)

... is the Fourier transform of the

The unit step function $U(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

$$h(t) = \begin{cases} 0, & t < 0 \\ e^{-\beta t}, & t \geq 0 \end{cases}$$

$$\therefore H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\beta t} e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(\beta + i\omega)t}\, dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} \left[e^{-(\beta+i\omega)t} \right]_0^{\infty}$$

$$= -\frac{1}{\beta+i\omega} [0-1]$$

$$= \frac{1}{\beta+i\omega}$$

$$|H(\omega)| = \frac{1}{|\beta+i\omega|} = \frac{1}{\sqrt{\beta^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2+\omega^2}$$

$$(1) \Rightarrow S_{yy}(\omega) = \frac{1}{\beta^2+\omega^2} S_{xx}(\omega).$$

Auto-correlation function of response:

$$R_{yy}(\tau) = R_{xx}(\tau) * h(-\tau) * h(\tau)$$

(4) Cross correlation functions of input and output

(i) $R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)$

Proof:

The cross correlation fun: of $x(t)$ and $y(t)$ is

$$R_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)] \quad \text{---} \ (1)$$

Now, $y(t+\tau) = h(t) * x(t+\tau)$

WKT

$$= \int_{-\infty}^{\infty} h(\varepsilon)\, x(t+\tau-\varepsilon)\, d\varepsilon \quad \text{---} \ (2)$$

Sub (2) in (1)

$$R_{xy}(t, t+\tau) = E\left[x(t)\int_{-\infty}^{\infty} h(\varepsilon)\, x(t+\tau-\varepsilon)\, d\varepsilon\right]$$

$$= \int_{-\infty}^{\infty} E[x(t)\, x(t+\tau-\varepsilon)]\, h(\varepsilon)\, d\varepsilon \quad \text{---} \ (3)$$

If $x(t)$ is WSS, (3) becomes

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-\varepsilon)\, h(\varepsilon)\, d\varepsilon$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h(\tau)$$

111ly $R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau)$.

(5) Consider a system with transfer function $\frac{1}{1+i\omega}$. An input signal with auto correlation function $m\,\delta(\tau) + m^2$ is fed as input to the system. Find the mean and mean square value of the input.

Soln

Gn $H(\omega) = \frac{1}{1+i\omega}$

& $R_{xx}(\tau) = m\,\delta(\tau) + m^2$

$$S_x(\omega) = m + 2\pi m^2 \delta(\omega)$$

WKT $S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$

$$= \left|\frac{1}{(1+i\omega)}\right|^2 \left[m + 2\pi m^2 \delta(\omega)\right]$$

WKT $S_{yy}(\omega) = S_{xx}(\omega)\,|H(\omega)|^2$ ——— ①

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|}\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau}\, d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{(2-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \left[\left(\frac{1}{2-i\omega}\right) - 0\right] + \left[0 - \frac{1}{-(2+i\omega)}\right]$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega}$$

$$= \frac{2+i\omega+2-i\omega}{4+\omega^2}$$

$$S_{xx}(\omega) = \frac{4}{2^2+\omega^2}$$

$$H(\omega) = \frac{1}{\omega+2i} = \frac{1}{\omega+2i} \times \frac{\omega-2i}{\omega-2i} = \frac{\omega-2i}{\omega^2+2^2}$$

$$|H(\omega)| = \sqrt{\left(\frac{\omega}{\omega^2+4}\right)^2 + \left(\frac{2}{\omega^2+4}\right)^2} = \sqrt{\frac{\omega^2+4}{(\omega^2+4)^2}} = \frac{1}{\sqrt{\omega^2+4}}$$

$$① \Rightarrow S_{yy}(\omega) = \left(\frac{4}{4+\omega^2}\right)\left(\frac{1}{\omega^2+4}\right) = \frac{4}{(\omega^2+4)^2}$$

__White Noise :__

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signals in ... we have

Shot Noise:

The discrete nature of electrons causes a signal disturbance called Shot noise

Thermal Noise:

This noise is due to the random motion of free electrons in a conducting Medium such as a resistor.

White Noise (or) Gaussian Noise

The Noise analysis of communication Systems is based on an idealized form of noise called white noise

Power Spectral density of thermal Noise:

The power Spectral density of the noise current due to the free electrons is given by

$$S_i(\omega) = \left[\frac{2kTG\alpha^2}{\alpha^2+\omega^2}\right] = \frac{2kTG}{1+\left(\frac{W}{\alpha}\right)^2}$$

where K is the Boltzmann's constant
α is the average number of collision / second
T is the ambient temperature in degrees kelvin
G is the conductance of the conducting medium

Band limited white Noise:

Noise having a non-zero and constant spectral density over a finite frequency band and zero elsewhere is called band limited white Noise (i.e) if $\{N(t)\}$ is a band limited white noise then

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2} & , |W| \leq W_B \\ 0 & , \text{elsewhere} \end{cases}$$

(7) Consider a white Gaussian noise of zero Mean and power spectral density $N_0/2$ applied ... RC filter where transfer function

is $H(f) = \frac{1}{1+i2\pi fRC}$. find the autocorrelation function of the output random process.

Soln

Gn $H(f) = \frac{1}{1+i2\pi fRC}$

$$|H(f)| = \frac{1}{|1+i2\pi fRC|} = \frac{1}{\sqrt{1+4\pi^2 f^2 R^2 C^2}}$$

$$|H(f)|^2 = \frac{1}{1+4\pi^2 f^2 R^2 C^2} \quad \text{——— (1)}$$

Gn $S_{XX}(f) = \frac{N_0}{2}$ ——— (2) [$\because$ the input is a white noise]

The Power Spectral densities of the input $\{X(t)\}$ and the output $\{Y(t)\}$ of a linear system are connected by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega) \quad \text{——— (3)}$$

In the given problem the transfer function is expressed interms of the frequency f

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$

$$= \frac{1}{1+4\pi^2 f^2 R^2 C^2} \frac{N_0}{2} \quad [\text{by (1) + (2)}]$$

$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(f)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2 R^2 C^2} \frac{N_0}{2}\, e^{i2\pi f\tau}\, d(2\pi f)$$

[$\because \omega = 2\pi f$]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2 R^2 C^2}\, e^{i2\pi\tau f}\, 2\pi\, df$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2 R^2 C^2}\, e^{i2\pi\tau f}\, df$$

$$R_{yy}(\tau) = \frac{N_0}{8\pi^2 R^2 C^2} \int_{-\infty}^{\infty} \frac{e^{i(2\pi\tau)f}}{\left(\frac{1}{2\pi RC}\right)^2 + f^2} \, df \quad \text{—— (4)}$$

$$= \frac{N_0}{8\pi^2 R^2 C^2} \frac{\pi}{\left(\frac{1}{2\pi RC}\right)} e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)} \qquad \left[\because \int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2} dx = \frac{\pi}{a} e^{-|m|a}\right]$$

$$= \frac{N_0}{8\pi^2 R^2 C^2} 2\pi^2 RC \; e^{-|2\pi\tau|\left(\frac{1}{2\pi RC}\right)}$$

$$R_{yy}(\tau) = \frac{N_0}{4RC} e^{-|\tau|/RC}$$

The Mean Square Value of $\{Y(t)\}$ is given by

$$E[Y^2(t)] = R_{yy}(0)$$

$$= \frac{N_0}{4RC} e^{-\frac{0}{RC}} = \frac{N_0}{4RC} e^{-0} = \frac{N_0}{4RC}(1)$$

$$= \frac{N_0}{4RC}$$

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(8) If $\{X(t)\}$ is a band limited process such that $S_{XX}(\omega) = 0$, when $|\omega| > \sigma$, prove that $2[R_{XX}(0) - R_{XX}(\tau)] \le \sigma^2\tau^2 R_{XX}(0)$.

Soln:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\tau\omega} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) (\cos\omega\tau + i\sin\omega\tau) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) \cos\omega\tau \, d\omega$$

$$R_{XX}(0) - R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau) d\omega \qquad [\because X(t) \text{ is band limited}]$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega) \, 2\sin^2\left(\frac{\omega\tau}{2}\right) d\omega \quad \text{—— (1)}$$

WKT $|\sin\theta| \le \theta$

$$2\sin^2\left(\frac{\omega \tau}{2}\right) \le \frac{\tau^2\omega^2}{2} \quad \text{——— (2)}$$

$$R_{XX}(0) - R_{XX}(\tau) \le \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\frac{\tau^2\omega^2}{2}\,d\omega$$

[by (1)]

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\sigma^2\tau^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$\le \frac{\sigma^2\tau^2}{\cancel{4}\pi}\cdot \cancel{2}\pi R_{XX}(0)$$

$$\le \frac{\sigma^2\tau^2}{2} R_{XX}(0).$$

(9) If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant, θ is a random variable with a uniform distribution in $(-\pi, \pi)$ and $\{N(t)\}$ is a band limited Gaussian white noise with a power spectral density.

$$S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & \text{for } |\omega - \omega_0| < \omega_B \\ 0, & \text{elsewhere} \end{cases}$$

find the power spectral density of $\{Y(t)\}$. Assume that $N(t)$ and θ are in independent.

Soln

Gn $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$$Y(t+\tau) = A\cos(\omega_0(t+\tau) + \theta) + N(t+\tau)$$

$$= A\cos[\omega_0 t + \omega_0\tau + \theta] + N(t+\tau)$$

$$Y(t)\,Y(t+\tau) = [A\cos(\omega_0 t + \theta) + N(t)][A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)]$$

$$= A^2\cos(\omega_0 t + \theta)\cos(\omega_0 t + \omega_0\tau + \theta)$$

$$+ A\cos(\omega_0 t + \theta)N(t+\tau) + A\cos(\omega_0 t + \omega_0\tau + \theta)N(t)$$

$$\therefore R_{YY}(t, t+\tau) = E[Y(t)\,Y(t+\tau)]$$

$$= E[A^2 \cos(\omega_0 + \theta) \cos(\omega_0 t + \omega_0 \tau + \theta) + A \cos(\omega_0 t + \omega_0\tau+\theta) N(t)$$
$$+ A \cos(\omega_0 t + \theta)\, N(t+\tau) + N(t)\, N(t+\tau)]$$

$$= A^2\, E[\cos(\omega_0 t+\theta)\cos(\omega_0 t + \omega_0\tau+\theta)] + A E[\cos(\omega_0 t+\theta) N(t+\tau)] + A E[\cos(\omega_0 t+\omega_0\tau+\theta) N(t)]$$
$$+ E[N(t)\, N(t+\tau)]$$

$$= \frac{A^2}{2} E[2\cos(\omega_0 t+\omega_0\tau+\theta)\cos(\omega_0 t+\theta)]$$
$$+ A\, E[\cos(\omega_0 t+\theta)\, N(t+\tau)]$$
$$+ A\, E[\cos(\omega_0 t+\omega_0\tau+\theta)\, N(t)]$$
$$+ E[N(t)\, N(t+\tau)]$$

$$= \frac{A^2}{2} E[\cos(\omega_0 t+\omega_0\tau+\theta+\omega_0 t+\theta) + \cos(\omega_0 t+\omega_0\tau+\theta-\omega_0 t-\theta)]$$
$$+ A\, E[\cos(\omega_0 t+\theta)\, N(t+\tau)]$$
$$+ A\, E[\cos(\omega_0 t+\omega_0\tau+\theta)\, N(t)]$$
$$+ E[N(t)\, N(t+\tau)]$$

$$= \frac{A^2}{2} E[\cos(2\omega_0 t+2\theta+\omega_0\tau) + \cos\omega_0\tau]$$
$$+ A\, E[\cos(\omega_0 t+\theta)]\, E(N(t+\tau))$$
$$+ A\, E[\cos(\omega_0 t+\omega_0\tau+\theta)]\, E[N(t)]$$
$$+ R_{NN}(\tau) \qquad [\because N(t) \text{ is stationary}] \quad \text{—— } (1)$$

As θ is uniformly distributed in $(-\pi, \pi)$

$$\therefore f(\theta) = \frac{1}{2\pi}, \quad -\pi < \theta < \pi$$

$$E[\cos(\omega_0 t+\theta)] = \int_{-\infty}^{\infty} \cos(\omega_0 t+\theta)\, f(\theta)\, d\theta$$
$$= \int_{-\pi}^{\pi} \cos(\omega_0 t+\theta)\, \frac{1}{2\pi}\, d\theta$$
$$= \frac{1}{2\pi}\int^{\pi} [\cos\omega_0 t \cos\theta - \sin\omega_0 t \sin\theta]\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\omega_0 t \cos\theta \, d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\omega_0 t \sin\theta \, d\theta$$

$$= \frac{1}{2\pi} \cos\omega_0 t \int_{-\pi}^{\pi} \cos\theta \, d\theta - \frac{1}{2\pi} \sin\omega_0 t \int_{-\pi}^{\pi} \sin\theta \, d\theta$$

$$= \frac{1}{2\pi} \cos\omega_0 t \quad 2\int_{0}^{\pi} \cos\theta \, d\theta - \frac{1}{2\pi} \sin\omega_0 t \int_{-\pi}^{\pi} \sin\theta \, d\theta$$

$$= \frac{1}{2\pi} \cos\omega_0 t \quad (2)\int_{0}^{\pi} \cos\theta \, d\theta - \frac{1}{2\pi} \sin\omega_0 t \; (0)$$

$$= \frac{1}{\pi} \cos\omega_0 t \left[\sin\theta\right]_0^{\pi} - 0$$

$$= \frac{1}{\pi} \cos\omega_0 t \left[0 - 0\right]$$

$$= \frac{1}{\pi}$$

$$= 0 \quad \text{——} \quad (2)$$

$$E[\cos(\omega_0 t + \omega_0 \tau + \theta)] = 0 \quad \text{——} \quad (3)$$

$$E[\cos(2\omega_0 t + 2\theta + 2\omega_0 \tau)] = \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta + \omega_0 \tau) \frac{1}{2\pi} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta + \omega_0 \tau) \, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\omega_0 t + \omega_0 \tau) \cos 2\theta - \sin(2\omega_0 t + \omega_0 \tau) \sin 2\theta \, d\theta$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau) \int_{-\pi}^{\pi} \cos 2\theta \, d\theta - \frac{1}{2\pi} \sin(2\omega_0 t + \omega_0 \tau) \int_{-\pi}^{\pi} \sin 2\theta \, d\theta$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau) \; 2\int_{0}^{\pi} \cos 2\theta \, d\theta - \frac{1}{2\pi} \sin(2\omega_0 t + \omega_0 \tau) (0)$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau) \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau) \left[\sin 2\theta\right]_0^{\pi}$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau) \left[0 - 0\right]$$

$$= \frac{1}{2\pi} \cos(2\omega_0 t + \omega_0 \tau)$$

$$= 0 \quad \text{——} \quad (4)$$

$$(1) \Rightarrow R_{YY}(t, t+\tau) = \frac{A^2}{2} \cos(\omega_0 \tau) + R_{NN}(\tau)$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} \left[\frac{A^2}{2} \cos \omega_0 \tau + R_{NN}(\tau) \right] e^{-i\omega\tau} d\tau$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} \cos \omega_0 \tau \, e^{-i\omega\tau} d\tau + \int_{-\infty}^{\infty} R_{NN}(\tau) \, e^{-i\omega\tau} d\tau$$

$$= \pi \frac{A^2}{2} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right] + S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2} \left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right] + \frac{N_0}{2}$$

$$\left[\because \text{Gn } S_{NN}(\omega) = \frac{N_0}{2} \right]$$