UNIT-IV

Correlation and Spectral densities:

Autocorrelation:

If the process $X(t)$ is either wide sense stationary (or) Strict Sense stationary then $E[X(t) \cdot X(t+\tau)]$ is a function of τ, denoded by $R(\tau)$ or $R_{xx}(\tau)$ or $R_x(\tau)$.

This fn. $R_{xx}(\tau)$ is called the autocorrelation fn. of the process $X(t)$.

(ie) $R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$

Properties:

Property 1: The mean square value of the random process may be obtained from the auto correlation fn. $R_{xx}(\tau)$, by putting $\tau = 0$.

Proof:

WKT

$$R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$$

$$R_{xx}(0) = E[X(t) \cdot X(t)]$$

$$= E[X^2(t)]$$

(ie) $R_{xx}(0)$ is the mean square value. (ie) II moment of the random process.

Property 2:

$R_{xx}(\tau)$ is an even fn. of τ.

(ie) $R_{xx}(\tau) = R_{xx}(-\tau)$.

Proof:

WKT

$$R_{xx}(\tau) = E\left[x(t).\, x(t+\tau)\right]$$

$$R_{xx}(-\tau) = E\left[x(t).\, x(t-\tau)\right]$$

Put $(t-\tau) = p$

$$R_{xx}(-\tau) = E\left[x(p+\tau) \cdot x(p)\right]$$

$$= R_{xx}(\tau)$$

$$\therefore R_{xx}(-\tau) = R_{xx}(\tau).$$

Property 3:

The maximum value of $R_{xx}(\tau)$ is obtained at the point $\tau = 0$, (ie)

$|R_{xx}(\tau)| \leq R_{xx}(0)$

Proof:

Consider $E\left\{\left[x(t_1) \pm x(t_2)\right]^2\right\} \geq 0$

$$E\left[x^2(t_1) + x^2(t_2) \pm 2\,x(t_1)\,x(t_2)\right] \geq 0$$

$$E\left[x^2(t_1)\right] + E\left[x^2(t_2)\right] \pm 2\,E\left[x(t_1) \cdot x(t_2)\right] \geq 0$$

By property 1,

$$R_{xx}(0) + R_{xx}(0) \pm 2\,R_{xx}(t_1, t_2) \geq 0$$

$$2R_{xx}(0) \geq |2R_{xx}(\tau)|$$

Property 4:

If a random process X(t) has no periodic components and it X(t) is of non-zero mean, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = [E[X]]^2$

Property 5:

If X(t) is periodic then its autocorrelation fn.. is also periodic.

Property 6:

If the random process $Z(t) = X(t) + Y(t)$. Where, X(t) and Y(t) are random process then $R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + R_{XY}(\tau) + R_{YX}(\tau)$.

1) Given that the autocorrelation fn.. for a Stationary ergodic process with no periodic components is $R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$. Find the mean and Variance of the process X(t).

Solution:

Given

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

By using property 4:

$$(E[X])^2 = \bar{X}^2 = \lim_{\tau\to\infty} R_{XX}(\tau)$$

$$= \lim_{\tau\to\infty} 25 + \frac{4}{1+6\tau^2}$$

$$= 25 + \frac{4}{\infty}$$

$$= 25+0$$

$\therefore E[x(t)] = 5$

By property 1,

$$E[x^2(t)] = R_{xx}(0)$$

$$= 25 + \frac{4}{1+6(0)}$$

$$= 25 + 4$$

$$= 29$$

$$E[x^2(t)] = 29$$

$$Var[x(t)] = E[x^2(t)] - E([x(t)])^2$$

$$= 29 - 25$$

$$= 4$$

$$\therefore Var(x(t)) = 4$$

2. Find the mean and variance of a stationary process whose autocorrelation function is $R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}$

3. Check whether the following fn's are valid autocorrelatio fnr.

$$R_{xx}(\tau) = \frac{25\tau^2}{4+5\tau^2}$$

$$R_{xx}(\tau) = \tau^3 + \tau^2$$

$$R_{xx}(\tau) = \cos\tau + \frac{|\tau|}{T}$$

2. Solution:

Given:

$$R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}$$

By using Property 4,

$$(E[x])^2 = \bar{X}^2 = \lim_{\tau \to \infty} R_{xx}(\tau)$$

$$= \lim_{\tau \to \infty} 18 + \frac{2}{6+\tau^2}$$

$$= 18 + \frac{2}{6+\infty}$$

$$= 18 + \frac{2}{\infty}$$

$$= 18 + 0$$

$$= 18$$

$$\bar{X}^2 = 18$$

$$\bar{X} = 4.24.$$

$$E[x(t)] = 4.24$$

By property 1:

$$E[x^2(t)] = R_{xx}(0)$$

$$= \cancel{25 + \frac{4}{1+}} = 18 + \frac{2}{6+0}$$

$$= 18 + \frac{2}{6}$$

$$= 18.33$$

$$E[x^2(t)] = 18.33$$

$$Var[x(t)] = E[x^2(t)] - (E[x(t)])^2$$

$$= 18.33 - 18$$

$$= 0.33$$

$$Var[x(t)] = 0.33$$

3. Solution:

Given:

(i) $R_{xx}(\tau) = \dfrac{25\tau^2}{4+5\tau^2}$

$R_{xx}(-\tau) = \dfrac{25(-\tau)^2}{4+5(-\tau)^2}$

$= \dfrac{25\tau^2}{4+5\tau^2}$

$\therefore R_{xx}(\tau) = R_{xx}(-\tau)$

$\therefore R_{xx}(\tau)$ is a autocorrelation fn.

(ii) $R_{xx}(\tau) = \tau^3+\tau^2$

$R_{xx}(-\tau) = (-\tau)^3+(-\tau)^2$

$= -\tau^3+\tau^2 \qquad R_{xx}(\tau) \neq R_{xx}(-\tau)$.

$R_{xx}(\tau)$ is not a autocorrelation fn.

(iii) $R_{xx}(\tau) = \cos\tau + \dfrac{|\tau|}{T}$

$R_{xx}(-\tau) = \cos(-\tau) + \dfrac{|-\tau|}{T}$

$= \cos\tau + \dfrac{\tau}{T}$

$R_{xx}(\tau) = R_{xx}(-\tau)$.

$R_{xx}(\tau)$ is a autocorrelation fn.

4. Show that, a random process $X(t) = A\sin(\omega t+\varphi)$ where A and ω are constants, φ is a random variable uniformly distributed in $(0, 2\pi)$. Find the autocorrelation fn.. of the process.

Solution,

Given:

$$X(t) = A\sin(\omega t+\varphi).$$

φ is uniformly distributed in $(0, 2\pi)$

$$f(\varphi) = \frac{1}{b-a} = \frac{1}{2\pi - 0}$$

$$f(\varphi) = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t).\ x(t+\tau)]$$

$$= E[A\sin(\omega t+\varphi).\ A\sin(\omega t+\omega\tau+\varphi)]$$

$$= A^2 E[\sin(\omega t+\varphi).\ \sin(\omega t+\omega\tau+\varphi)]$$

$$= \frac{A^2}{2} E\left[\cos(-\omega\tau) - \cos(2\omega t+\omega\tau+2\varphi)\right]$$

$$= \frac{A^2}{2} E\left[\cos(\omega\tau) - \cos(2\omega t+\omega\tau+2\varphi)\right]$$

$$= \frac{A^2}{2} E[\cos(\omega\tau)] - \frac{A^2}{2} E[\cos(2\omega t+\omega\tau+2\varphi)]$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{2}\int_0^{2\pi} \cos(2\omega t+\omega\tau+2\varphi)\, f(\varphi)\, d\varphi.$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{2}\left[\frac{\sin(2\omega t+\omega\tau+2\varphi)}{2\cdot 2\pi}\right]_0^{2\pi} d\varphi.$$

$$= \frac{A^2}{2}\cos\omega\tau - \frac{A^2}{2}[0]$$

Cross correlation:

Let $X(t)$ and $Y(t)$ be two random processes, then the cross correlation b/w them is defined as

$$R_{XY}(t,t+\tau) = E[X(t) \cdot Y(t+\tau)] = R_{XY}(\tau)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y;t,t+\tau)\, dx\, dy.$$

Property 1:

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

Proof:

$$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$$

Consider,

$$R_{YX}(-\tau) = E[X(t-\tau)\ Y(t)]$$

Put $t-\tau = a \Rightarrow t = a+\tau$

$$R_{YX}(\tau) = E[X(a) \cdot Y(a+\tau)]$$

$$= R_{XY}(\tau).$$

Property 2:

If $X(t)$ and $Y(t)$ are two random process, $R_{XX}(\tau)$ and $R_{YY}(\tau)$ are their respective and correlation fn. then;

$$|R_{XY}(\tau)| \le \sqrt{R_{XX}(0)\ R_{YY}(0)}$$

Consider,

$$E\left\{\left[\frac{X(t)}{\sqrt{R_{XX}(0)}} - \frac{Y(t)}{\sqrt{R_{YY}(0)}}\right]^2\right\} \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)} + \frac{Y^2(t)}{R_{YY}(0)} - 2\frac{X(t).Y(t)}{\sqrt{R_{XX}(0)\ R_{YY}(0)}}\right] \geq 0$$

$$E\left[\frac{X^2(t)}{R_{XX}(0)}\right] + E\left[\frac{Y^2(t)}{R_{YY}(0)}\right] - 2E\left[\frac{X(t).Y(t)}{\sqrt{R_{XX}(0).R_{YY}(0)}}\right] \geq 0.$$

$$\frac{1}{R_{XX}(0)} E\left[X^2(t)\right] + \frac{1}{R_{YY}(0)} E\left[Y^2(t)\right] - \frac{2}{\sqrt{R_{XX}(0).R_{YY}(0)}} E\left[X(t).Y(t)\right] \geq 0$$

$$\frac{R_{XX}(0)}{R_{XX}(0)} + \frac{R_{YY}(0)}{R_{YY}(0)} - \frac{2}{\sqrt{R_{XX}(0).R_{YY}(0)}} R_{XY}(\tau) \geq 0.$$

$$1+1 - \frac{2}{\sqrt{R_{XX}(0) R_{YY}(0)}} . R_{XY}(\tau) \geq 0$$

$$2 \geq 2 . \frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0). R_{YY}0}}$$

$$\sqrt{R_{XX}(0). R_{YY}(0)} \geq |R_{XY}(\tau).|$$

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0). R_{YY}(0)}.$$

Hence proved.

Property 3:

If $X(t)$ and $Y(t)$ are two random process then,

$$|R_{XY}(\tau)| \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$$

Property 4:

If the random process $X(t)$ and $Y(t)$ are independent, then

$$R_{XY}(\tau) = E[X].E[Y].$$

$$R_{XY}(\tau) = E[X(t).Y(t+\tau)]$$

Property 5:

If the random process $X(t)$ and $Y(t)$ are of zero mean,

$$\lim_{\tau\to\infty} R_{XY}(\tau) = \lim_{\tau\to\infty} R_{YX}(\tau) = 0$$

Property 6:

The autocorrelation and cross correlation of two random processes $X(t)$ and $Y(t)$ can be expressed as a matrix called correlation matrix

$$R(\tau) = \begin{bmatrix} R_{XX}(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_{YY}(\tau) \end{bmatrix}$$

then the random process $X(t)$ and $Y(t)$ are jointly WSS process.

Property 7:

Two random processes $x(t)$ and $Y(t)$ are said to be uncorrelated, if their cross correlation fn. is equal to the product of their means.

$$R_{XY}(\tau) = E[x(t)] \cdot E[Y(t+\tau)^*]$$

5. Two random processes $X(t)$ and $Y(t)$ are given by $X(t) = A\cos(\omega t+\theta)$, $Y(t) = A\sin(\omega t+\theta)$ where, A and ω are constants and θ is a uniform random variable over $(0, 2\pi)$. Find the cross correlation fn.

Solution:

Given:

$$X(t) = A\cos(\omega t+\theta)$$

$$Y(t) = A\sin(\omega t+\theta)$$

θ is uniform random variable,

$$f(\theta) = \frac{1}{b-a} = \frac{1}{2\pi-0} = \frac{1}{2\pi}$$

$$f(\theta) = \frac{1}{2\pi}$$

$$R_{XY}(\tau) = E[x(t) \cdot Y(t+\tau)]$$

$$= E\left[A\cos(\omega t+\theta) \cdot A\sin\overset{(\omega t+\omega\tau+\theta)}{\cancel{(\omega t+\theta)}}\right]$$

$$= A^2 E\left[\cos(\omega t+\theta), \sin(\omega t+\omega\tau+\theta)\right]$$

$$= \frac{A^2}{}\left[\sin(2\omega t+\omega\tau+\theta) \cdot \cos(\omega t+\theta)\right]$$

$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta)+\sin(-\omega\tau)\right]$$

$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta)+\sin\omega\tau\right]$$

$$= \frac{A^2}{2} E\left[\sin(2\omega t+\omega\tau+2\theta)\right]+E\left[\sin\omega\tau\right]$$

$$= \frac{A^2}{2}\sin\omega\tau+\frac{A^2}{2}\int_0^{2\pi}\sin(2\omega t+\omega\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{A^2}{2}\sin\omega\tau+\frac{A^2}{4\pi}\left[\frac{-\cos(2\omega t+\omega\tau+2\theta)}{2}\right]_0^{2\pi}$$

$\cos(4\pi+\theta) = \cos(2\pi+\theta) = \cos(\theta)$

$$= \frac{A^2}{2}\sin\omega\tau+\frac{A^2}{8\pi}\left[-\cos(2\omega t+\omega\tau+4\pi)+\cos(2\omega t+\omega\tau)\right]$$

$$= \frac{A^2}{2}\sin\omega\tau+\frac{A^2}{8\pi}\left[\cancel{-\cos(2\omega t+\omega\tau)+\cos(2\omega t+\omega\tau)}\right]$$

$$= \frac{A^2}{2}\sin\omega\tau$$

3/4/2012

Power Spectral Density

The power spectral density $S_{x}(\omega)$ of a continuous time random process $X(t)$ is defined as the fourier transform of $R_{xx}(\tau)$, $S_{xx}(\omega) = -\infty$ to ∞

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-i\omega\tau} d\tau \quad \text{---(1)}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau} d\omega \quad \text{---(2)}$$

Eqn (1) and (2) are known as the Wiener-Khinchine relation.

Property :

(i) For a WSS, random process, prove spectral density at 0 (zero) frequency gives the area under the graph of autocorrelation.

(ii) The mean square value of a WSS process is equal to the total area under the graph of the spectral density.

(iii) The PSD of a real valued random process is an even fn.. of frequency (or) The spectral density fn.. of a real random process is an even fn..

(iv) A WSS, random process has a non-

(v) The Spectral density and the autocorrelation fn.. of a real WSS process form a fourier Cosine transform Pair.

6. The PSD of a WSS process is given by $S(\omega) = \begin{cases} b/a\,(a-|\omega|) & |\omega| \le a \\ 0 & |\omega| \ge a \end{cases}$

Find the autocorrelation fn.. of a process:

Solution:

Given:

$$S(\omega) = \begin{cases} \frac{b}{a}\left(a-|\omega|\right) & |\omega| \le a \\ 0 & |\omega| \ge a \end{cases}$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\, e^{i\omega\tau}\, d\omega$$

$$= 2\cdot\frac{1}{2\pi}\int_{0}^{a} \frac{b}{a}(a-\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{\pi}\,\frac{b}{a}\int_{0}^{a}(a-\omega)\, e^{i\omega\tau}\, d\omega.$$

$u = a-\omega \qquad v = e^{i\omega\tau}$

$u' = -\omega \qquad v_1 = \frac{e^{i\omega\tau}}{i\tau}$

$u'' = -1$

$u''' = 0 \qquad v_2 = \frac{e^{i\omega\tau}}{i^2\tau^2}$

$$= \frac{b}{\pi a}\int_0^a (a-\omega)(\cos\omega\tau + i\sin\omega\tau)\,d\omega$$

$$= \frac{b}{\pi a}\int_0^a (a-\omega)\cos\omega\tau \, . \, d\omega$$

$u = a-\omega \qquad V = \cos\omega\tau$

$u' = -\omega \qquad V_1 = \frac{\sin w\tau}{\tau}$

$u'' = -1 \qquad V_2 = -\frac{\cos\omega\tau}{\tau^2}$

$u = a-\omega \qquad V = \cos\omega\tau$

$u' = -1 \qquad V_1 = \frac{\sin\omega\tau}{\tau}$

$u'' = 0 \qquad V_2 = -\frac{\cos\omega\tau}{\tau^2}$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} - \frac{\cos\omega\tau}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left[\left[0 - \frac{\cos a\tau}{\tau^2}\right] - \left[\frac{a\sin(0)}{\tau} - \frac{\cos(0)}{\tau^2}\right]\right]$$

$$= \frac{b}{\pi a}\left[-\frac{\cos a\tau}{\tau^2} + \frac{1}{\tau^2}\right]$$

$$= \frac{b}{a\tau^2\pi}\left[1-\cos a\tau\right] \qquad 1-\cos\theta = 2\sin^2\theta/2$$

$$= \frac{b}{a\tau^2\pi}\, 2\sin^2 a\tau/2$$

7. The autocorrelation fn. of a WSS process is given by $R(\tau) = \alpha^2 e^{-2\lambda|\tau|}$. Determine the PSD of the process.

Solution:

Given, $R_{XX}(\tau) = \alpha^2 e^{-2\lambda|\tau|}$

Let $\alpha^2 = a$

$2\lambda = b$

$\therefore R_{XX}(\tau) = a e^{-b|\tau|}, \quad b>0$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} a e^{-b|\tau|} \cdot e^{-i\omega\tau}\, d\tau$$

In $(-\infty, 0) \Rightarrow |\tau| = -\tau$

$(0,\infty) \Rightarrow |\tau| = \tau$

$$S_{XX}(\omega) = \int_{-\infty}^{0} a e^{-b(-\tau)} e^{-i\omega\tau}\, d\tau + \int_{0}^{\infty} a e^{-b\tau} e^{-i\omega\tau}\, d\tau$$

$$= a\int_{-\infty}^{0} e^{b\tau} \cdot e^{-i\omega\tau}\, d\tau + a\int_{0}^{\infty} e^{-b\tau} \cdot e^{-i\omega\tau}\, d\tau$$

$$= a\int e^{(b-i\omega)\tau}\, d\tau + a\int e^{-(b+i\omega)\tau}\, d\tau$$

$$= a\left[\frac{e^{(b-i\omega)x}}{b-i\omega}\right]_{-\infty}^{0} + a\left[\frac{e^{-(b+i\omega)x}}{-(b+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{a}{b-i\omega}\left[e^{0} - e^{-\infty}\right] - \frac{a}{b+i\omega}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{a}{b-i\omega}\left[1-0\right] - \frac{a}{b+i\omega}\left[0-1\right]$$

$$= \frac{a}{b-i\omega} + \frac{a}{b+i\omega}$$

$$= a\left[\frac{1}{b-i\omega} + \frac{1}{b+i\omega}\right]$$

$$= a\left[\frac{b+i\cancel{\omega} + b-i\cancel{\omega}}{b^2+\omega^2}\right]$$

$$= a\left[\frac{2b}{b^2+\omega^2}\right]$$

$$= \frac{2ab}{b^2+\omega^2}$$

$$S_{xx}(\omega) = \frac{2ab}{b^2+\omega^2}$$

$$S_{xx}(\omega) = \frac{2\alpha^2 2\lambda}{(2\lambda)^2+\omega^2}$$

$$\boxed{S_{xx}(\omega) = \frac{4\alpha^2\lambda}{4\lambda^2+\omega^2}}$$

8. The PSD of a WSS process is given by $S(\omega) = \begin{cases} 1, & |\omega| \leq \omega_0 \\ 0 & \text{Otherwise.} \end{cases}$

Find the autocorrelation fn/- of a process.

Solution:-

Given,

$$S(\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{Otherwise} \end{cases}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \cdot e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\omega_0} (\cos\omega\tau + i\sin\omega\tau)\, d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\omega_0} \cos\omega\tau\, d\omega$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega\tau}{\tau}\right]_{0}^{\omega_0}$$

$$= \frac{1}{\pi}\left[\frac{\sin\omega_0\tau}{\tau} - \frac{\sin(0)\tau}{\tau}\right]$$

$$= \frac{1}{\pi}\,\frac{\sin\omega_0\tau}{\tau}$$

9. Find PSD for the stationary process $X(t)$ with autocorrelation fn. $R_{XX}(\tau) = a e^{-b|\tau|}$, $b>0$.
(or) $R_{XX}(\tau) = \sigma^2 e^{-\alpha|\tau|}$

10. The autocorrelation of the random binary transmission is given by $R_{XX}(\tau) = \begin{cases} 1-\frac{|\tau|}{T}, & |\tau| \le T \\ 0 & |\tau| \ge T \end{cases}$

Find PSD.

Solution:

Given:

$$R_{XX}(\tau) = \begin{cases} 1-\frac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| \ge T \end{cases}$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-T}^{T} 1 - \frac{|\tau|}{T} e^{-i\omega\tau} d\tau$$

$$= \int_{-T}^{T} 1 - \frac{|\tau|}{T} (\cos\omega\tau - i\sin\omega\tau) d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{\tau}{T}\right) \cos\omega\tau \, d\tau$$

$$= \int_{-T}^{T} \left(\cos\omega\tau - \frac{\tau}{T}\cos\omega\tau\right) d\tau$$

$$= 2\int^{T} \cos\omega\tau \, d\tau - 2\int^{T} \tau$$

$$= 2\left[\frac{\sin\omega\tau}{\omega}\right]_0^T - \frac{2}{T}\int_0^T \tau\cos\omega\tau \, d\tau$$

$$u = \tau \qquad v = \cos\omega\tau$$

$$u' = 1 \qquad v_1 = \frac{\sin\omega\tau}{\omega}$$

$$u'' = 0 \qquad v_2 = -\frac{\cos\omega\tau}{\omega^2}$$

$$= 2\left[\frac{\sin\omega T}{\omega} - \frac{\sin(0)}{\omega}\right] - \frac{2}{T}\left[\tau\frac{\sin\omega\tau}{\omega} + \frac{\cos\omega\tau}{\omega^2}\right]_0^T$$

$$= \frac{2\sin\omega T}{\omega} - \frac{2}{T}\left[T\frac{\sin\omega T}{\omega} + \frac{\cos\omega T}{\omega^2} - \frac{\cos 0}{\omega^2}\right]$$

$$= \cancel{\frac{2\sin\omega T}{\omega}} - \cancel{\frac{2\sin\omega T}{\omega}} - \frac{2\cos\omega T}{T\omega^2} + \frac{2}{T\omega^2}$$

$$= \frac{2}{T\omega^2}\left[1 - \cos\omega T\right]$$

CROSS SPECTRAL DENSITY.

Let $R_{XY}(\tau)$ and $R_{YX}(\tau)$ be their Cross correlation fn/.. then Cross Spectral densities are

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-i\omega\tau}\, d\tau.$$

Property 1:

$S_{YX}(\omega) = S_{XY}(-\omega)$

Property 2:

Real part of $S_{XY}(\omega)$ is an even fn/.. of ω

Property 3:

Imag part of $X_{XY}(\omega)$ is an odd fn/.. of ω

Property 4:

$S_{XY}(\omega) = 0$, if $X(t)$ ~~...~~ are orthogonal

Property 5:

If $X(t)$ and $Y(t)$ are uncorrelated,

$S_{XY}(\omega) = E[X].\, E[Y].\, \delta(\omega).$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \ldots\, e^{-i\omega\tau}$$

$X(t)$ and $Y(t)$ are said to be uncorrelated.

11. The cross power spectrum of real random processes $X(t)$ and $Y(t)$ is given by,

$$S_{XY}(\omega) = \begin{cases} a + \dfrac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha,\ \alpha > 0 \\ 0, & \text{Otherwise} \end{cases}$$

Find the cross correlation fn..

Solution:

Given

$$S_{XY}(\omega) = \begin{cases} a + \dfrac{ib\omega}{\alpha}, & -\alpha < \omega < \alpha,\ \alpha > 0 \\ 0, & \text{Otherwise} \end{cases}$$

Cross correlation fn. is

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left(a + \frac{ib\omega}{\alpha}\right) e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} a\, e^{i\omega\tau}\, d\omega + \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{ib\omega}{\alpha}\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\int_{-\alpha}^{\alpha} e^{i\omega\tau}\, d\omega + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha} \omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{2a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{0}^{\alpha} + \frac{2ib}{2\pi\alpha}\int_{0}^{\alpha} \omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2\pi\alpha}\int_{-\alpha}^{\alpha} \omega\, e^{i\omega\tau}\, d\omega$$

$$= \frac{a}{\tau 2\pi i}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha}\left[\omega \cdot \frac{e^{i\omega\tau}}{i\tau} + \frac{e^{i\omega\tau}}{\tau^2}\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{2\pi i\tau}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right] + \frac{ib}{2\pi\alpha}\left[\left[\alpha\frac{e^{i\alpha\tau}}{i\tau} + \frac{e^{i\alpha\tau}}{\tau^2}\right] - \left[-\alpha\frac{e^{-i\alpha\tau}}{i\tau} + \frac{e^{-i\alpha\tau}}{\tau^2}\right]\right]$$

$$= \frac{a}{2\pi i\tau}\left[2i\sin\alpha\tau\right] + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\left[e^{i\alpha\tau} + e^{-i\alpha\tau}\right] + \frac{1}{\tau^2}\left[e^{i\alpha\tau} - e^{-i\alpha\tau}\right]\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib}{2\pi\alpha}\left[\frac{\alpha}{i\tau}\, 2\cos\alpha\tau + \frac{1}{\tau^2}\, 2i\sin\alpha\tau\right]$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{ib\,2\alpha}{2\pi\alpha i\tau}\cos\alpha\tau + \frac{ib\,2i}{2\pi\alpha\tau^2}\sin\alpha\tau$$

$$R_{xx}(\tau) = \frac{a}{\pi\tau}\sin\alpha\tau + \frac{b}{\ldots}\cos\alpha\tau - \frac{b}{\ldots}\sin\ldots$$

4/4/13

12 If the cross correlation of two processes $X(t)$ and $Y(t)$ is $R_{XY}(t, t+\tau) = \frac{AB}{2}\left[\sin w_0\tau + \cos w_0(2t+\tau)\right]$ where A and B, w_0 are constants. Find the cross power spectrum and time average

Solution:

The time average is given by

$$P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left(\frac{AB}{2}\sin w_0\tau + \cos w_0(2t+\tau)\right)dt$$

$$\cancel{= \lim_{T\to\infty}\frac{AB}{2}\left[\lim_{T\to\infty}\int_{-T}^{T}\sin w_0\tau\, dt\right]}$$

$$= \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin w_0\tau\, dt +$$

$$\frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos w_0(2t+\tau)\, dt$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt +$$

$$\frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos w_0(2t+\tau)\, dt$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\left[T\right]_{-T}^{T} +$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\left[T+T\right] + 0$$

$$\cancel{\frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int \sin}$$

$$= \frac{AB}{2}\sin w_0\tau \lim_{T\to\infty}\frac{1}{2T}\cdot 2T$$

$$= \frac{AB}{2}\sin w_0\tau .$$

Cross Power Spectrum is

$$S_{XY}(\omega) = FT \text{ of } \frac{AB}{2}\sin w_0\tau$$

$$= \int_{-\infty}^{\infty}\frac{AB}{2}\sin w_0\tau \, e^{-iw\tau}\, d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty}\sin w_0\tau\left(\cos w\tau - i\sin w\tau\right) d\tau$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty}\sin w_0\tau \cos w_0\tau \, d\tau \; -$$

$$i\frac{AB}{2}\int_{-\infty}^{\infty}\sin w_0\tau \, . \, \sin w\tau \, d\tau .$$

$$= \frac{AB}{2}\int_{-\infty}^{\infty}\frac{1}{2}\sin(w_0+w)\tau + \sin(w_0+w)\tau \; -$$

$$i\frac{AB}{2}\int_{-\infty}^{\infty}\frac{1}{2}\cos(w_0-w)\tau - \cos(w_0+w)\tau$$

$$= \frac{AB}{4}\left[\int_{-\infty}^{\infty}\left[i\left(\cos(\omega+\omega_0)\tau - i\sin(\omega+\omega_0)\tau\right)\right.\right.$$

$$\left.\left. - i\left(\cos(\omega-\omega_0)\tau - i\sin(\omega-\omega_0)\tau\right)\right] d\tau\right.$$

$\sin(-\theta) = \sin\theta$

$$= \frac{ABi}{4}\int_{-\infty}^{\infty}\left(e^{-i(\omega+\omega_0)\tau} - e^{-i(\omega-\omega_0)\tau}\right) d\tau$$

$$= \frac{-iAB}{4}\int_{-\infty}^{\infty}\left(e^{-i(\omega-\omega_0)\tau} - e^{-i(\omega+\omega_0)\tau}\right) d\tau$$

$$= -\frac{iAB}{4}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau}\, d\tau + \frac{iAB}{4}\int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau}\, d\tau.$$

$$= -\frac{iAB}{4}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right]$$

$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau$ is the

dirac - delta fn/.. Such that

$$\int_{-\infty}^{\infty}\delta(\omega)\, d\omega = 1$$

$$\therefore\ S_{XY}(\omega) = -\frac{i\pi AB}{2}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

13 If $X(t)$ and $Y(t)$ are uncorrelated random process, then find the power spectral density of Z, if $Z(t) = X(t)+Y(t)$. Also find the cross spectral density $S_{XZ}(\omega)$ and $S_{YZ}(\omega)$

Solution:

If $X(t)$ and $Y(t)$ are uncorrelated random process, their cross covariance $C_{XY}(t, t+\tau) = 0$

$$R_{XY}(t, t+\tau) - E[X(t)]\,E[Y(t)] = 0$$

$$R_{XY}[t, t+\tau] = E[X(t)] \cdot E[Y(t)]$$

$$R_{XY}(\tau) = E[X(t)] \cdot E[Y(t)].$$

Illy

$$R_{YX}(\tau) = E[Y(t)] \cdot E[X(t)]$$

$$= R_{XY}(\tau)$$

$$Z(t) = X(t) + Y(t).$$

$$R_{ZZ}(\tau) = E[Z(t) \cdot Z(t+\tau)]$$

$$= E\big[[X(t)+Y(t)][X(t+\tau) + Y(t+\tau)]\big]$$

$$= E\big[X(t) \cdot X(t+\tau) + X(t) \cdot Y(t+\tau) + Y(t) \cdot X(t+\tau) + Y(t) \cdot Y(t+\tau)\big]$$

$$= E[X(t) \cdot X(t+\tau)] + E[X(t) \cdot Y(t+\tau)] + E[Y(t) \cdot X(t+\tau)] + E[Y(t) \cdot Y(t+\tau)]$$

$$S_{ZZ}(\omega) = \int_{-\infty}^{\infty} R_{ZZ}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(R_{XX}(\tau) + 2R_{XY}(\tau) + R_{YY}(\tau)\right) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} 2R_{XY}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$+ \int_{-\infty}^{\infty} R_{YY}(\tau)\, e^{-i\omega\tau}\, d\tau.$$

$$S_{ZZ}(\omega) = S_{XX}(\omega) + 2S_{XY}(\omega) + S_{YY}(\omega)$$

Cross correlation function:

$$R_{XZ}(\tau) = E\left[X(t).\, Z(t+\tau)\right]$$

$$= E\left[X(t).\, X(t+\tau) + Y(t+\tau)\right]$$

$$= E\left[X(t).X(t+\tau) + X(t).Y(t+\tau)\right]$$

$$= E\left[X(t).X(t+\tau)\right] + E\left[X(t).\, Y(t+\tau)\right]$$

$$R_{XZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau).$$

$$S_{XZ}(\omega) = \int_{-\infty}^{\infty} R_{XZ}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \left(R_{XX}(\tau) + R_{XY}(\tau)\right) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= S_{XX}(\omega) + S_{XY}(\omega)$$

$$S_{XZ}(\omega) = S_{XX}(\omega) + S_{XY}(\omega)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int E\left[|X_T(\omega)|^2\right] = S_{XX}(\omega)$$

$$|||^{ly} \quad S_{YZ}(\omega) = S_{YY}(\omega) + S_{YX}(\omega)$$

Find the mean square value of the process, PSD is b/w $S_{XX}(\omega) = \frac{\omega^2+2}{\omega^4+13\omega^2+36}$

Solution!

Given

$$S_{XX}(\omega) = \frac{\omega^2+2}{\omega^4+13\omega^2+36} \qquad a=1 \quad b=13 \quad c=36$$

$$= \frac{\omega^2+2}{(\omega^2)^2+13\omega^2+36}$$

$$= \frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)}$$

By partial fraction method.

$$\frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)} = \frac{A}{\omega^2+9} + \frac{B}{\omega^2+4}$$

$$\frac{\omega^2+2}{(\omega^2+9)\cancel{(\omega^2+4)}} = \frac{A(\omega^2+4)+B(\omega^2+9)}{(\omega^2+9)\cancel{(\omega^2+4)}}$$

$$\omega^2+2 = A(\omega^2+4) + B(\omega^2+9)$$

$$-2 = 5B$$

$$\boxed{B = -2/5}$$

Put $\omega^2 = -9$

$$-9+2 = A(-9+4) + B(-9+9)$$

$$-7 = A(-5)$$

$$7 = 5A$$

$$\boxed{A = 7/5}$$

$$\therefore \frac{\omega^2+2}{(\omega^2+9)(\omega^2+4)} = \frac{7/5}{\omega^2+9} - \frac{2/5}{\omega^2+4}$$

$$S_{xx}(\omega) = \frac{7/5}{\omega^2+9} - \frac{2/5}{\omega^2+4}$$

$$R_{xx}(\tau) = F^{-1}\left[S_{xx}(\omega)\right]$$

$$= F^{-1}\left[\frac{7}{5}\cdot\frac{1}{\omega^2+9} - \frac{2}{5}\,\frac{1}{\omega^2+4}\right]$$

$$= \frac{7}{5}F^{-1}\left[\frac{1}{\omega^2+9}\right] - \frac{2}{5}F^{-1}\left[\frac{1}{\omega^2+4}\right]$$

$$\left[\because F^{-1}\left[\frac{2\alpha}{\omega^2+\alpha^2}\right] = e^{-\alpha|\tau|}\right]$$

$$= \frac{7}{5}F^{-1}\left[\frac{1}{6}\cdot\frac{2.3}{\omega^2+3^2}\right] - \frac{2}{5}F^{-1}\left[\frac{1}{4}\cdot\frac{2.2}{\omega^2+4}\right]$$

$$= \frac{7}{30}F^{-1}\left[\frac{6}{\omega^2+3^2}\right] - \frac{2}{20}F^{-1}\left[\frac{4}{\omega^2+2^2}\right]$$

$$R_{xx}(\tau). = \frac{7}{\ }e^{-3|\tau|} - \frac{1}{\ }e^{-2|\tau|}$$

$$E[x^2(t)] = R_{xx}(0)$$

$$R_{xx}(0) = \frac{7}{30} e^{-3(0)} - \frac{1}{10} e^{-2(0)}$$

$$= \frac{7}{30} - \frac{1}{10}$$

$$= \frac{7-3}{30}$$

$$= \frac{4}{30}$$

$$= 2/15$$

$$E[x^2(t)] = 2/15$$