UNIT-III

Classification of Random Processes.

Stationary process:

Random Variable:

A random variable is a rule that assigns a real number to every outcome of a random experiment.

Random process.

A random process is a rule that assigns a time function to every outcome of a random experiment.

A random process is a collection of random variable $\{X(s,t)\}$ $s \in S$ (Sample space) $t \in T$ (Parameter set).

Classification of Random processes.

Discrete Random Sequence.

If both S and T are discrete then the random process is called discrete random sequence.

Eg. No. of books in library at opening time.

Continuous Random Sequence.

If 'S' is continuous and T is discrete, then the random process is

Eg: Quantity of petrol in the bulk at opening time

Discrete Random process

If 'S' is discrete and 'T' is Continuous and the random process is called discrete random process.

Eg: No. of phone calls receiving in $(0,t)$

Continuous Random Variable Process

If 'S' is continuous and T is continuous, then the random process is called continuous random process

Eg. Stirring sugar in coffee

Strick Sense Stationary:

A random process is called a strongly stationary process (or) Strict sense stationary process (SSS) ib all its finite dimensional distributions are invariant under translation of time parameter.

Note:

$x(t)$ is SSS

ib (i) $E[x(t)]$ is constant

(ii) $E[x^2(t)]$ is Constant.

[illegible] is Constant

Wide Sense Stationary:

A random process is called wide sense stationary (WSS) (or) Weakly Stationary process (or) Covariance stationary process.

if (i) $E[X(t)]$ is constant

(ii) Auto correlation is a function of τ (free from 't') (WSS).

Note:

A random process, non-stationary is called an evolutionary process.

$X(t)$ and $Y(t)$ are said to be Jointly WSS

(i) $R_{XY}(\tau)$ is a function of τ.

if (ii) each process is individually WSS.

22.2.2013.

1. Show that it is not-stationary. The process $X(t)$ whose probability distribution under certain conditions is given by

$$P\{X(t)=n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n=1,2,\dots \\ \dfrac{at}{1+at} & n=0 \end{cases}$$

Solution:

$X(t)=n$	0	1	2	....
[illegible]	at	1	[illegible]	

(ii) $E[x^2(t)] = \sum_{n=0}^{\infty} n^2 p(n)$

$$= 0 + \sum_{n=1}^{\infty} n^2 \left(\frac{(at)^{n-1}}{(1+at)^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n^2 (at)^{n-1}}{(1+at)^{n-1+2}}$$

$$= \frac{1}{(1+at)^2} \sum_{n=1}^{\infty} \{n(n+1) - n\} \left(\frac{at}{1+at}\right)^{n-1}$$

$$= \frac{1}{(1+at)^2} \left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$= \frac{1}{(1+at)^2}\left[\left[1\cdot 2\left(\frac{at}{1+at}\right)^0 + 2\cdot 3\left(\frac{at}{1+at}\right)^1 + \cdots\right] - \left[1\left(\frac{at}{1+at}\right)^0 + 2\left(\frac{at}{1+at}\right)^1 + 3\left(\frac{at}{1+at}\right)^2 + \cdots\right]\right]$$

$$= \frac{1}{(1+at)^2}\left[\left[2\left(1 + 3\left(\frac{at}{1+at}\right) + 6\left(\frac{at}{1+at}\right)^2 + \cdots\right] - \left[1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \cdots\right]\right]$$

$$= \frac{1}{(1+at)^2}\left[2\left(1 - \frac{at}{1+at}\right)^{-3} - \left(1 - \frac{at}{1+at}\right)^{-2}\right]$$

$$= \frac{1}{(1+at)^2}\left[2\left(\frac{1+at-at}{1+at}\right)^{-3} - \left(\frac{1+at-at}{1+at}\right)^{-2}\right]$$

Given,

$$X(t) = \cos(\lambda t + Y)$$

$$\varphi(\omega) = E[\cos \omega Y + i \sin \omega Y]$$

$\varphi(1) = 0$

$$\Rightarrow \varphi(1) = E[\cos Y + i \sin Y] = 0$$

$$E[\cos Y + i \sin Y] = 0.$$

$$E[\cos Y] + i\, E[\sin Y] = 0$$

$$E[\cos Y] = 0 \quad \& \quad E[\sin Y] = 0$$

$\varphi(2) = 0$

$$\Rightarrow E[\cos 2Y + i \sin 2Y] = 0$$

$$E[\cos 2Y] + iE[\sin 2Y] = 0$$

$$E[\cos 2Y] = 0 \quad \& \quad E[\sin 2Y] = 0.$$

(i) $E[X(t)] = E[\cos(\lambda t + Y)]$

$$= E[\cos(\lambda t)\cos y - \sin \lambda t \sin y]$$

$$= \cos \lambda t\, E[\cos Y] + \sin \lambda t\, E[\sin Y]$$

$$= 0, \text{ constant} \cdot \because E[\cos Y] = 0, \; E[\sin Y] = 0$$

(ii) $R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$

$$= E[\cos(\lambda t + y) \cdot \cos(\lambda(t+\tau) + \cdots$$

$$= E\left[\cos\lambda t \; \cos\lambda(t+\tau) \; \cos^2 y + \sin\lambda t \; \sin\lambda(t+\tau)\; \sin^2 y\right] - \left[\cos\lambda t \; \sin\lambda(t+\tau)\; \cos y \sin y + \sin\lambda t \; \cos\lambda(t+\tau)\; \cos y \sin y\right]$$

$$= \cos\lambda t \; \cos\lambda(t+\lambda)\; E[\cos^2 y] + \sin\lambda t \; \sin\lambda(t+\tau)\; E[\sin^2 y] + \left[\cos\lambda t \; \sin\lambda(t+\tau) + \sin\lambda t \; \cos\lambda(t+\tau)\right] E\left[\cos y \sin y\right]$$

$$= \cos\lambda t \; \cos\lambda(t+\tau)\; E\left[\frac{1+\cos 2y}{2}\right] + \sin\lambda t \; \sin\lambda(t+\tau)\; E\left[\frac{1-\cos 2y}{2}\right] + E\left[\frac{\sin 2y}{2}\right]\left[\cos\lambda t \; \sin\lambda(t+\tau) + \sin\lambda t \; \cos\lambda(t+\tau)\right]$$

$$= \frac{1}{2}\left[\cos\lambda t \; \cos\lambda(t+\tau)\right] + \frac{1}{2}\left[\sin\lambda t \; \sin\lambda(t+\tau)\right]$$

$$= \frac{1}{2}\left[\cos(\lambda t - \lambda(t+\tau))\right]$$

$$= \frac{1}{2}\left[\cos(\lambda t - \lambda t - \lambda\tau)\right]$$

$$= \frac{1}{2}\cos\lambda\tau \text{ , free from } t.$$

$X(t)$ is WSS.

4. Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$.

(A, B) are random variables. is WSS.

(i) $E(A] = E[B] = 0$

(ii) $E[A^2] = E[B^2] = \varphi$

(iii) $E[AB] = 0$.

(i) $E[X(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= \cos\lambda t\, E[A] + \sin\lambda t\, E[B]$.

$= 0$, constant.

(ii) $R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$

$= E\Big[[A\cos\lambda t + B\sin\lambda t]$

$(A\cos(\lambda(t+\tau)) + B\sin(\lambda(t+\tau)))\Big]$

$= E\Big[A^2\cos\lambda t\ \cos\lambda(t+\tau) + B^2\sin\lambda t\ \sin\lambda(t+\tau)$

$+ AB\cos\lambda t\sin\lambda(t+\tau) +$

$AB\sin\lambda t\ \cos\lambda(t+\tau)\Big]$

$= \cos\lambda t\ \cos\lambda(t+\tau)\, E[A^2] +$

$\sin\lambda t.\ \sin\lambda(t+\tau)\, E[B^2] +$

$E[AB]\Big[\cos\lambda t\ \sin\lambda(t+\tau) + \sin\lambda t\ \sin\lambda(t+\tau)\Big]$

$= \cos\lambda t\ \cos\lambda(t+\tau)\,\varphi +$

$= \varphi \left[\cos \lambda t \cos \lambda(t+\tau) + \sin \lambda t \sin \lambda(t+\tau) \right]$

$= \varphi \left[\cos(\lambda t - (\lambda(t+\tau)) \right]$

$= \varphi \left[\cos(\lambda t - \lambda t - \lambda \tau \right]$

$= \varphi \cos \lambda \tau$, free from t,

$X(t)$ is WSS.

5. Two random processes $X(t)$ and $Y(t)$ are given by

$X(t) = A \cos \omega t + B \sin \omega t$

$Y(t) = B \cos \omega t - A \sin \omega t$.

Show that, $X(t)$ and $Y(t)$ are jointly WSS, If A and B are uncorrelated R.V with zero mean and the same variances with and ω is constant.

Solution:

$X(t) = A \cos \omega t + B \sin \omega t$

$Y(t) = B \cos \omega t - A \sin \omega t$.

$E[A] = E[B] = 0$.

$Var(A) = Var(B) = \sigma^2$

$\Rightarrow E[A^2] = E[B^2] = \sigma^2$

A and B are uncorrelated $\Rightarrow E[AB] = 0$.

(i) $X(t)$ is WSS

$$E[X(t)] = E[A\cos\omega t + B\sin\omega t]$$

$$= E[A]\cos\omega t + E[B]\sin\omega t$$

$$= 0,\ \text{Constant}$$

$$R_{xx}(\tau) = E[X(t)\cdot X(t+\tau)]$$

$$= E\left[(A\cos\omega t + B\sin\omega t)\,(A\cos(\omega(t+\tau)) + B\sin(\omega(t+\tau)))\right]$$

$$= E\Big[A^2\cos\omega t\cos(\omega(t+\tau)) + B^2\sin\omega t\sin(\omega(t+\tau))$$

$$+ AB\left[\cos\omega t\sin(\omega(t+\tau)) + \sin\omega t\cos(\omega(t+\tau))\right]\Big]$$

$$= E[A^2]\cos\omega t\cos(\omega(t+\tau)) + E[B^2]\sin\omega t\sin(\omega(t+\tau))$$

$$+ E[AB]\left[\cos\omega t\sin(\omega(t+\tau)) + \sin\omega t\cos(\omega(t+\tau))\right]$$

$$= \sigma^2\left[\cos\omega t\cos(\omega(t+\tau)) + \sin\omega t\sin(\omega(t+\tau))\right] + 0$$

$$= \sigma^2\left(\cos(\omega t - \omega(t+\tau))\right)$$

$$= \sigma^2\left(\cos(\omega t - \omega t - \omega\tau)\right)$$

$$= \sigma^2\left(\cos(-\omega\tau)\right)$$

$$= \sigma^2\cos\omega\tau,\ \text{free from } t,$$

$X(t)$ is WSS.

(ii) $Y(t)$ is WSS.

$$E[Y(t)] = E[B \cancel{\cos} \omega t - A \sin \omega t]$$

$$= E[B] \cos \omega t - E[A] \sin \omega t$$

$$= 0, \text{ constant},$$

$$R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$$

$$= E\Big[(B \cos \omega t - A \sin \omega t)(B \cos(\omega(t+\tau)) - A \sin(\omega(t+\tau))\Big]$$

$$= E\Big[B^2 \cos \omega t \cos(\omega(t+\tau)) + A^2 \sin \omega t \sin(\omega(t+\tau)) - AB \cos \omega t \sin(\omega(t+\tau)) - AB \sin \omega t \cancel{\sin} \cos(\omega(t+\tau))\Big]$$

$$= E[B^2] \cos \omega t \cos(\omega(t+\tau)) + E[A^2] \sin \omega t \sin(\omega(t+\tau)) - E[AB]\Big[\cos \omega t \sin(\omega(t+\tau)) + \sin \omega t \cos(\omega(t+\tau))\Big]$$

$$= \sigma^2 \Big[\cos \omega t \cos(\omega(t+\tau)) + \sin \omega t \sin(\omega(t+\tau))\Big]$$

$$= \sigma^2 \big(\cos(\omega t - \omega(t+\tau))\big]$$

$$= \sigma^2 \big(\cos(\cancel{\omega t} - \cancel{\omega t} - \omega \tau)\big]$$

$$= \sigma^2 [\cos(-\omega \tau)]$$

$$= \sigma^2 \cos \omega \tau$$

$R_{YY}(t) = \sigma^2 \cos \omega \tau$, free from t.

$R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

$= E[(A\cos\omega t + B\sin\omega t)(B\cos(\omega(t+\tau)) - A\sin(\omega(t+\tau)))]$

$= E[B^2\sin\omega t \cos(\omega(t+\tau)) - A^2\cos\omega t \sin(\omega(t+\tau))$

$+ AB\cos\omega t \cos(\omega(t+\tau)) - AB(\sin\omega t \sin(\omega(t+\tau))]$

$= \sigma^2(\sin\omega t \cos(\omega(t+\tau)) - \cos\omega t \sin\omega(t+\tau))$

$+ 0$

$= \sigma^2[\sin(\omega t - \omega t - \omega\tau)]$

$= \sigma^2[\sin(-\omega\tau)]$

$= -\sigma^2 \sin\omega\tau$, free from t,

b. If $X(t) = Y\cos t + Z\sin t$ $\forall t$, where Y and Z are independent binary random variables each of which assumes the values $(-1, +2)$ with probabilities $2/3$ and $1/3$ respectively. Prove that $X(t)$ is WSS.

Solution:

Given

$X(t) = Y\cos t + Z\sin t$

Y	-1	2
$P(Y)$	$2/3$	$1/3$

Z	-1	2
$P(Z)$	$2/3$	$1/3$

$= E[Y^2] \cos t \cos(t+\tau) + E[Z^2] \sin t \sin(t+\tau)$

$+ E[YZ][\cos t \sin(t+\tau) + \sin t \cos(t+\tau)]$

$= 2[\cos t \cos(t+\tau) + \sin t \sin(t+\tau)] + 0$

$= 2(\cos(t - t - \tau))$

$= 2\cos(-\tau)$

$= 2\cos\tau$. free from t.

$X(t)$ is WSS.

25/2/2013

Ergodicity:

A random process $X(t)$ is said to be ergodic, if its ensemble averages (statistical averages (ie) mean, autocorrelation) are equal to appropriate time averages.

If $X(t)$ is a random process, then $\frac{1}{2T}\int_{-T}^{T} X(t)\,dt$ is called time average of $X(t)$ over $(-T, T)$ and denoted by $\overline{X}_T$

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt.$$

If the random process $X(t)$ has a constant mean, as $T \to \infty$, then $x(t)$ is said to be mean ergodic.

Problem procedure:

Step 1: Find $\overline{x_T}$

Step 2: Find $E[\overline{x_T}]_T$

Step 3: $Var(\overline{x_T}) = \frac{1}{T}\int C_{xx}(\tau)\left(1-\frac{|\tau|}{T}\right)d\tau$

where

$C_{xx}(\tau) = E[x(t)\ x(t+\tau)] - E[x(t)]E[x(t+\tau)]$

Step 4: $\lim_{T\to\infty} Var(\overline{x_T}) = 0$

Correlation ergodic:

$x(t)$ is correlation ergodic,

if $\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\ x(t)\,dt = R(\tau)$

as limit $T \to \infty$

7. If WSS process $x(t)$ is given by $x(t) = 10\cos(100t+\theta)$ where θ is uniformly distributed over $(-\pi, \pi)$. Prove that $x(t)$ is correlation ergodic.

Solution:

$$f(\theta) = \frac{1}{b-a} = \frac{1}{\pi+\pi} = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t)\cdot x(t+\tau)]$$

$$= E[10\cos(100t+\theta)\cdot 10\cos(100(t+\tau)+\theta)]$$

$$= \frac{100}{2} E\left[\cos(200t+100\tau+2\theta)+\cos(100\tau)\right]$$

$$= 50\, E\left[\cos(200t+100\tau+2\theta)\right]+E\left[\cos(100\tau)\right]$$

Consider,

$$E\left[\cos(200t+100\tau+2\theta)\right]$$

$$= \int_{-\pi}^{\pi} \cos(200t+100\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta$$

$$= \frac{1}{2\pi}\cdot 2\int_{0}^{\pi} \cos(200t+100\tau+2\theta)\,d\theta$$

$$= \frac{1}{\pi}\left[\frac{\sin(200t+100\tau+2\theta)\cdot 2}{2}\right]_{0}^{\pi}$$

$$= 0$$

Sub in (1)

$$R_{xx}(\tau) = 50\{0+\cos 100\tau\}$$

$$= 50\cos 100\tau$$

$$\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T} X(t+\tau)\,X(t)\,dt = R(\tau)$$

$$= \frac{1}{2T}\int_{-T}^{T} 10\cos(100t+\theta)\,10\cos(100(t+\tau)+\theta)\,dt$$

$$= \frac{100}{2T}\int_{-T}^{T} \cos(100t+\theta)\cdot\cos(100t+100\tau+\theta)\,dt$$

$$= \frac{50}{T}\int_{-T}^{T}\frac{1}{2}\left[\cos(200t+100\tau+2\theta)+\cos(-100\tau)\right]$$

$Var\ \bar{x}_T = 2/3$

$\lim_{T\to\infty} Var\ \bar{x}_T = \frac{2}{3} \neq 0$

$X(t)$ is not mean ergodic.

Consider 2 random ~~variable~~ process,
$X(t) = 3\cos(\omega t+\theta)$ $\quad Y(t) = 2\cos(\omega t+\theta-\pi/2)$
where θ is a random variable uniformly distributed in $(0, 2\pi)$. Prove that $\sqrt{R_{xx}(0)\cdot R_{yy}(0)} \geq |R_{xy}(\tau)|$

Solution,

Given.

$X(t) = 3\cos(\omega t+\theta)$

$Y(t) = 2\cos(\omega t+\theta-\pi/2)$

θ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \frac{1}{b-a} = \frac{1}{2\pi-0} = \frac{1}{2\pi}$$

$$R_{xx}(\tau) = E[x(t)\cdot x(t+\tau)]$$

$$= E\left[3\cos(\omega t+\theta)\cdot 3\cos(\omega(t+\tau)+\theta)\right]$$

$$= 9E\left[\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta)\right]$$

$$= \frac{9}{2}E\left[\cos(2\omega t+\omega\tau+2\theta)+\cos(-\omega\tau)\right]$$

$= \frac{9}{[illegible]} E[\cos(2 [illegible]$

$$= \frac{9}{2} E\left[\cos(2\omega t + \omega\tau + 2\theta)\right] + \frac{9}{2} E\left[\cos(\cancel{2}\omega\tau)\right]$$

$$= \frac{9}{2}\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta)\cdot\frac{1}{2\pi}\, d\theta + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{4\pi}\underbrace{\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2}\right]_0^{2\pi}}_{0} + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{2}\cos\omega\tau$$

$$R_{XX}(0) = \frac{9}{2}\cos\omega(0)$$

$$= \frac{9}{2}$$

$$R_{YY}(\tau) = E\left[Y(t)\cdot Y(t+\tau)\right]$$

$$= E\left[2\cos(\omega t + \theta - \pi/2)\cdot 2\cos(\omega t + \omega\tau + \theta - \pi/2)\right]$$

$$= \cancel{2}\, 4E\left[\cos(\omega t + \theta - \pi/2)\cdot\cos(\omega t + \omega\tau + \theta - \pi/2)\right]$$

$$= \frac{4}{2}E\left[\cos(2\omega t + \omega\tau + 2\theta - \pi) + \cos(-\omega\tau)\right]$$

$$= 2E\left[\cos(2\omega t + \omega\tau + 2\theta - \pi)\right] + 2E\left[\cos(\omega\tau)\right]$$

$$= 2\int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta + \pi)\frac{1}{2\pi}\, d\theta + 2\cos\omega\tau$$

$$= 2\left[\sin(2\omega t + \omega\tau + 2\theta + \pi)\right]^{2\pi}$$

$$= 2\cos\omega\tau$$

$$R_{YY}(0) = 2\cos\omega(0)$$

$$= 2$$

$$R_{XY}(\tau) = E\left[X(t)\cdot Y(t+\tau)\right]$$

$$= E\left[3\cos(\omega t+\theta)\cdot 2\cos(\omega t+\omega\tau+\theta-\pi/2)\right]$$

$$= 6E\left[\cos(\omega t+\theta)\cdot\cos(\omega t+\omega\tau+\theta-\pi/2)\right]$$

$$= 6E\left[\cos(2\omega t+\omega\tau+2\theta-\pi/2)+\cos(-\omega\tau+\pi/2)\right]$$

$$= 6E\left[\cos(2\omega t+\omega\tau+2\theta-\pi/2)\right] +$$

$$6E\left[\cos\left(\overset{\pi/2-\omega\tau}{\cancel{\omega\tau-\pi/2}}\right)\right]$$

$$= 6\int_0^{2\pi}\cos(2\omega t+\omega\tau+2\theta-\pi/2)\cdot\frac{1}{2\pi}\,d\theta$$

$$+6\cos\left(\overset{\pi/2+\omega\tau}{\cancel{\omega\tau-\pi/2}}\right)$$

$$= 6E\left[\cos(\omega t+\theta)\cdot\sin(\omega t+\omega\tau+\theta)\right]$$

$$= \frac{6}{2}E\left[\sin(2\omega t+\omega\tau+2\theta)+\sin(\omega\tau)\right]$$

$$= \frac{6}{2}E\left[\sin(2\omega t+\omega\tau+2\theta)\right]+\frac{6}{2}E\left[\sin\omega\tau\right]$$

$$= \frac{6}{2}\int_0^{2\pi}\sin(2\omega t+\omega\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta+\frac{6}{2}\sin\omega\tau$$

$$= \frac{6}{2}\left[\frac{\cos(2\omega t+\omega\tau+2\theta)}{2\cdot 2\pi}\right]_0^{2\pi} + \frac{6}{2}\sin\omega\tau$$

$$= \frac{6}{2}\left[\frac{\cos(2\omega t+\omega\tau+2\pi)}{2\quad 2\pi} - \cos(0)\right] + \frac{6}{2}\sin\omega\tau$$

$$= 3\sin\omega\tau$$

$$R_{XY}(\tau) = 3\sin\omega\tau$$

$$R_{XX}(0)\cdot R_{YY}(0) = \frac{9}{2}\cdot 2 = 9$$

$$\sqrt{R_{XX}(0)\cdot R_{YY}(0)} = \sqrt{9}$$

$$= 3$$

$$R_{XY}(\tau) = |3\sin\omega\tau| \le 3$$

$$R_{XY}(\tau) \le \sqrt{R_{XX}(0)\cdot R_{YY}(0)}$$

Markov Process

Future depends only upon the Present but not on past.

If for all n, $P[x_n = a_n / x_{n-1} = a_{n-1}$

$P[x_n = a_n / x_{n-1} = a_{n-1} \cdots x_0 = a_0] =$

$$P[x_n = a_n / x_{n-1} = a_{n-1}]$$

$\{x_n\}$ then the process $\{x_n\}$, $n = 0, 1, 2 \ldots$ is called Markov Chain.

(i) $a_1, a_1, \cdots a_n$ are called States.

(ii) $P[x_n = a_j / x_{n-1} = a_i]$ is called one step

(iii) $P[x_n = a_j / x_0 = a_i]$ is called 'n' step transistion probability from State a_i to a_j.

Note 1:

The tpm of a margkov Change is a Stochastic matrix since $P_{ij} \geq 0$ and $\Sigma P_{ij} = 1$ (ie) Sum of all the elements of row of the

Note 2:

A Stochastic matrix 'p' is said to be a regular matrix, if all the entries of P^m (Possible integer m) are Positive.

Note 3:

A homogeneous markov chain is said to be regular, if its tpm is regular

Note 4:

If $P_{ij}^{m} > 0$, for some 'n' and $\forall$ i and j, that every state can be reached from every other state. Here, markov chain is said to be irreducable.

Note 5:

The period di of a return state, i is defined as the greatest common divisor of all m, such that $P_{ij}^{m} > 0$. State i is said to be periodic with period di, if $di > 1$, and aperiodic if $di = 1$

Note 6:

A non-null persistent and Apeariodic state is called ergodic.

Note 7:

If a markov chain is irreducible, all its states are of the same time.

if a markov chain is finite irreducible, all its states are non-null persistant.

Note 8:

Steady state, probability distribution or Stationary state distribution of the markov chain is $\pi p = \pi$

Note 9:

To find irreducible nature; $P^2, P^3, P^4 \ldots$ and note all $P_{ij} > 0$, at some p^n.

To find period type; called the Powers of P, where $P_{ii} > 0$, and find gcd of powers

To find steady state; find $\pi p = \pi$

Find the nature of the states of the markov chain if the tpm

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1/2 & 0 & 1/2 \\ 2 & 0 & 1 & 0 \end{array}$$

→ future. ↓ Present

Solution,

Given.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Markov chain is introducable and finite.

$\Rightarrow$ All states are non-null Persistant.

i) $P_{11}^{(2)} > 0$, $P_{11}^{(4)} > 0$

$\Rightarrow gcd\{2, 4 \ldots\} = 2$

$\Rightarrow$ state (0 is Period 2)

$P_{22}^{(2)} > 0$, $P_{22}^{(4)} > 0$

$gcd\{2, 4, \ldots\} = 2$

$\Rightarrow$ state 1 is Period 2

$P_{33}^{(2)} > 0$, $P_{33}^{(4)} > 0 \ldots$

$gcd\{2, 4 \ldots\} = 2$

$\Rightarrow$ State 2 is period 2.

Here all states are periodic with period 2.

Here all states are non-null persistent and periodic

$\Rightarrow$ All states are ergodic

Three Boys A, B, C are throwing a ball to each other, A always throws the ball to B. and B always throws the ball to C. But C is just as likely to throw the Ball to B, as to A. Find the tpm and classify the states.

Solution:

The tpm is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P^2 \cdot P^2 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P^2 \cdot P^3 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

The markov chain is introducable & finite

All states non-null persistant

$P_{11}^{(3)} > 0$, $P_{11}^{(5)} > 0$

$\gcd\{3, 5 \ldots\} = 1$

states 'A' is Period 1.

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0 \ldots$

$\gcd\{2, 3, 4, \ldots\} = 1$

state 'B' is Period 1

$P_{33}^{(2)} > 0$, $P_{33}^{(3)} > 0$, $P_{33}^{(4)} > 0 \ldots$

$\gcd\{2, 3, 4 \ldots\} = 1$

states 'C' is period 1

All states are Aperiodic

All states are ergodic.

A man drives a car (or) cadges a train to go to office each day. He never goes two days in a row by train but if the drives one day then the next day he is just likely to drive again as he is travel by train. Now suppose that the first day of the week. The ... drove to

The markov chain is introducable & finite

All states non-null persistant

$P_{11}^{(3)} > 0$, $P_{11}^{(5)} > 0$

$gcd\{3, 5 \dots\} = 1$

states 'A' is Period 1.

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0 \dots$

$gcd\{2, 3, 4, \dots\} = 1$

state 'B' is Period 1

$P_{33}^{(2)} > 0$, $P_{33}^{(3)} > 0$, $P_{33}^{(4)} > 0 \dots$

$gcd\{2, 3, 4 \dots\} = 1$

states 'C' is period 1

All states are Aperiodic

All states are ergodic.

A man drives a car (or) cadges a train to go to office each day. He never goes two days in a row by train but if the drives one day then the next day he is just likely to drive again as he is travel by train. Now suppose that the first day of the weak. The the first day ... drove to

The transition probability matrix of a Markov chain $\{X_n\} = 1, 2, 3$ having three states 1, 2 and 3 is $P=$

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ and the initial distribution is $P^{(0)} = [0.7, \ 0.2 \ \ 0.1]$

Find (i) $P[X_2 = 3]$

(ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Solution:

Given:

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$P^2 = P.P = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(i) $P[X_2 = 3] = \sum_{i=1}^{3} \left[P[X_2 = 3 / X_0 = i] . P[X_0 = i] \right]$

$= P[X_2 = 3 / X_0 = 1] . P[X_0 = 1] +$

$P[X_2 = 3 / X_0 = 2] \ P[X_0 = 2] +$

$P[X_2 = 3 / X_0 = 3] \ P[X_0 = 3]$

$= P_{13}^{(2)} \ P[X_0 = 1] + P_{23}^{(2)} \ P[X_0 = 2] +$

$P_{33}^{(2)} \ P[X_0 = 3]$

$= (0.26 \times 0.7) + (0.34 \times 0.2) +$

$= 0.182 + 0.068 + 0.029$

$= 0.279.$

(ii) $P[X_3=2, X_2=3, X_1=3, X_0=2] =$

$P[X_3=2/X_2=3, X_1=3, X_0=2]\, P[X_2=3, X_1=3, X_0=2]$

$= P[X_3=2/X_2=3]\, P[X_2=3 / X_1=3, X_0=2]\, P[X_1=3, X_0=2]$

$= P[X_3=2/X_2=3]\, P[X_2=3 / X_1=3]\, P[X_1=3 / X_0=2]\, P[X_0=2]$

$= P_{32}^{(1)} \cdot P_{33}^{(1)}\, P_{23}^{(1)} \cdot P[X_0=2]$

$= (0.4)(0.3)(0.2)(0.2)$

$= 0.0048.$

A State i is said to be recurrent, if the returned to State i, is certain, $F_{ii}=1$, (or) Transient.

if the return to State i is uncertain

$F_{ii} < 1$

Consider a markov chain with a States $\{0,1\}$. and $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

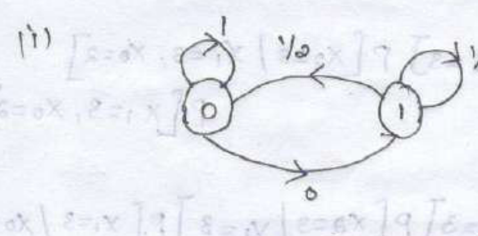
~~Solut~~ (i) Draw the tpm diagram.

(ii) Is the State - 0 recurrent?

(iii) Is the State - 1 is transient?

Solution:

(i)



(ii) State - 0 is recurrent

It returns to zero with probability 1,

(iii) State - 1 is Transient.

It returns to 1 with probability 1/2

Poisson process:

If $x(t)$ represents the number of occurence of a certain event in $(0,t)$, then discrete random process $x(t)$ is called the poisson process.

(i) $P(1 \text{ occurrence in } (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

(ii) $P[0 \text{ occurrence in } (t, t+\Delta t)] = (1-\lambda\Delta t) + o(\Delta t)$

(iii) $P[2 \text{ occurrence in } (t, t+\Delta t)] = o(\Delta t)$.

(iv) $x(t)$ is independent

$$P_n(t) = \left[\frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, 2 \ldots\right]$$

Second Order Probabity function of a homogeneous poisson process.

$$P[x(t_1)=n_1, x(t_2)=n_2] = P[x(t_1)=n_1]\, P[x(t_2)=n_2 \,/\, x(t_1)=n_1]$$

$$P[x(t_1)=n_1, x(t_2)=n_2] = \frac{e^{-\lambda t_1}(\lambda t_1)^{n_1}}{n_1!} \cdot \frac{e^{-\lambda(t_2-t_1)}\{\lambda(t_2-t_1)\}^{n_2-n_1}}{(n_2-n_1)!}$$

$n_2 \geq n_1$

Third Order probability function of a homogeneous poisson process:

$$P[x(t_1)=n_1, x(t_2)=n_2, x(t_3)=n_3] =$$

$$\frac{e^{-\lambda t_3}\,\lambda^{n_3}\cdot t_1^{n_1}(t_2-t_1)^{n_2-n_1}\cdot(t_3-t_2)^{n_3-n_2}}{n_1!\,(n_2-n_1)!\,(n_3-n_2)!} \quad n_3 \geq n_2 \geq n_1$$

Mean of a poisson process:

$$\text{Mean} = E[x(t)] = \sum_{n=0}^{\infty} n\, P_n(t)$$

$$= \sum_{n=0}^{\infty} n \cdot e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

$$= 0 + \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!}$$

$$= e^{-\lambda t}\left[\lambda(t) \quad (\lambda t)^2 \cdot (\lambda t)^3 \quad \right]$$

Auto co-variance of the poisson process:

$$C_{xx}(t_1,t_2)=R_{xx}(t_1,t_2)-E[x(t_1)].E[x(t_2)]$$

$$=\lambda^2(t_1t_2)+\lambda t_1-\lambda t_1 . \lambda t_2$$

$$=\lambda^2(\cancel{t_1}t_2)+\lambda(t_1)-\lambda^2(\cancel{t_1}t_2)$$

$$=\lambda(t_1)$$

$$\boxed{C_{xx}(t_1,t_2)=\lambda \min(t_1,t_2)}$$

Correlation Coefficient of the poisson process:.

$$\rho_{xx}(t_1,t_2)=\frac{C_{xx}(t_1,t_2)}{\sqrt{Var(x(t_1))}.\sqrt{Var(x(t_2))}}$$

$$=\frac{\lambda(t_1)}{\sqrt{\lambda t_1}.\sqrt{\lambda t_2}}$$

$$=\frac{\cancel{\lambda}t_1}{\cancel{\lambda}\sqrt{t_1t_2}}$$

$$=\sqrt{t_1/t_2} \quad t_1\le t_2.$$

$$\boxed{\rho_{xx}(t_1,t_2)=\sqrt{\frac{t_1}{t_2}}, \quad t_1\le t_2}$$

Property 1:

Poisson process is a markov process:

Let us take the conditional Probability distribution of $X(t_3)$ given the Past values of $X(t_2)$ and $X(t_1)$. Assume that $t_3 > t_2 > t_1$ and $n_3 > n_2 > n_1$

Consider $P[X(t_3) = n_3 / X(t_2) = n_2,\ X(t_1) = n_1]$

$$= \frac{P[X(t_3) = n_3,\ X(t_2) = n_2,\ X(t_1) = n_1]}{P[X(t_1) = n_1,\ X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda t_3} \cdot \lambda^{n_3} \cdot t_1^{n_1} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{n_1!\,(n_2 - n_1)!\,(n_3 - n_2)!} \div \frac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!\,(n_2 - n_1)!}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} \cdot \cancel{t_1^{n_1}} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{\cancel{n_1!}\,\cancel{(n_2 - n_1)!}\,(n_3 - n_2)!} \times \frac{\cancel{n_1!}\,\cancel{(n_2 - n_1)!}}{e^{-\lambda t_2} \lambda^{n_2} \cancel{t_1^{n_1}} \cancel{(t_2 - t_1)^{n_2 - n_1}}}$$

$$= \frac{e^{-\lambda t_3} \lambda^{n_3} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!\, e^{-\lambda t_2} \lambda^{n_2}}$$

$$= \frac{e^{-\lambda t_3} \cdot \lambda^{n_3} (t_3 - t_2)^{n_3 - n_2} e^{\lambda t_2} \lambda^{-n_2}}{(n_3 - n_2)!}$$

$$= \frac{e^{-\lambda(t_3-t_2)} \lambda^{(n_3-n_2)} (t_3-t_2)^{n_3-n_2}}{(n_3-n_2)!}$$

$\therefore$ Thus poisson process is a markov process.

Property 2:

Sum of two independent poisson Process is a poisson process.

Let $X(t) = X_1(t) + X_2(t)$.

$$P[X(t)=n] = \sum_{r=0}^{n} P[X_1(t)=r]\, P[X_2(t)=n-r]$$

$$= \sum_{r=0}^{n} e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^r}{r!} \cdot e^{-\lambda_2 t} \frac{(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \left[\sum_{r=0}^{n} \frac{(\lambda_1 t)^r}{r!} \cdot \frac{(\lambda_2 t)^{n-r}}{(n-r)!} \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} \left[\sum_{r=0}^{n} \frac{\lambda_1^r t^r}{r!} \times \frac{\lambda_2^{n-r} \cdot t^{n-r}}{(n-r)!} \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{n} \frac{t^n \cdot n!}{n!\, r!\, (n-r)!} \cdot \lambda_1^r \lambda_2^{n-r}$$

$$= e^{-(\lambda+\lambda_2)t} \sum_{r=0}^{n} \frac{t^n}{n!} nC_r\, \lambda_1^r \lambda_2^{n-r}$$

$$= e^{-(\lambda_1+\lambda_2)t} \cdot t^n \sum \lambda_1^r \lambda_2^{n-r} \cdot nC_r$$

$$\left(nc_x\, p^x q^{n-x}\right) = (p+q)^n$$

$$\therefore P[x(t)=n] = e^{-(\lambda_1+\lambda_2)t} \frac{t^n}{n!} (\lambda+\lambda_2)^n$$

$$= e^{-(\lambda_1+\lambda_2)t} \cdot \frac{[(\lambda_1+\lambda_2)t]^n}{n!}$$

Thus $X_1(t) + X_2(t)$ is a poisson process.

13/3/2013

Property 3:

Difference of two independant poisson process is not a poisson process.

Proof:

Let $x(t) = x_1(t) - x_2(t)$.

$$E[x(t)] = E[x_1(t) - x_2(t)]$$

$$= \lambda_1 t - \lambda_2 t$$

$$E[x(t)] = (\lambda_1 - \lambda_2)t$$

$$E[x^2(t)] = E\left[[x_1(t) - x_2(t)]^2\right]$$

$$= E\left[x_1^2(t) + x_2^2(t) - 2x_1(t) \cdot x_2(t)\right]$$

$$= E[x_1^2(t)] + E[x_2^2(t)] - 2E[x_1(t)] \cdot E[x_2(t)]$$

$$= (\lambda_1^2 t^2 + \lambda_1 t) + (\lambda_2^2 t^2 + \lambda_2 t) - 2\lambda_1 t \lambda_2 t$$

$$= (\lambda_1+\lambda_2)t + (\lambda_1^2+\lambda_2^2 - 2\lambda_1\lambda_2 t)^2$$

$$= (\lambda_1+\lambda_2)t + (\lambda_1-\lambda_2)^2 t^2$$

$[X_1(t) - X_2(t)]$ is not a poisson process.

Property 4:

The inter arrival time of a poisson process

The interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $1/\lambda$.

Proof:

Let E_i and E_{i+1} be the two consecutive events

Let T be the interval b/w E_i & E_{i+1}

$P[T>t] = P$[no event occurs in the interval length 't']

$$= P[X(t) = 0] \qquad P[X(t)=n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \frac{e^{-\lambda t}(\lambda t)^0}{0!}$$

$$= e^{-\lambda t}$$

$$F(t) = P(T \le t) = 1 - P(T>t)$$

$$f(t) = F'(t) = -e^{-\lambda t}(-\lambda)$$
$$= \lambda e^{-\lambda t}$$

$f(t) = \lambda e^{-\lambda t}$ is a pdf of an exponential distribution with mean $1/\lambda$

If the no. of occurrences of an event E in an interval of length t, is a Poisson process X(t) with parameter λ and if each occurrence of E has a constant probability P being recorded and the recordings are independent of each other, then the ~~no.~~ number N(t) of the recorded occurrences in 't' is also a poisson process with parameter λp

Solution:

$P[N(t)]$

$$P[N(t)=n] = \sum_{r=0}^{\infty} P\left[E \text{ occurs } (n+r) \text{ times in } t \text{ and } n \text{ of them are recorded}\right]$$

$$= \sum_{r=0}^{\infty} \left\{ \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \right\} \left\{ (n+r)C_n \, p^n q^{(n+r)-n} \right\}$$

$$= \sum_{r=0}^{\infty} \left\{ \frac{e^{-\lambda t} \lambda^n \lambda^r t^n t^r}{(n+r)!} \frac{(n+r)!}{(n+r-r)!\,r!} p^n q^{n+r-n} \right\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t} \lambda^r \lambda^n t^n t^r}{\cancel{(n+r)!}} \cdot \frac{\cancel{(n+r)!}}{n!\, r!} p^n \cdot q^r$$

$$= \frac{e^{-\lambda t} (\lambda t p)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda t q)^r}{r!}$$

$$= e^{-\lambda t} \frac{(\lambda t p)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda t q)^r}{r!}$$

$$= e^{-\lambda t} \frac{(\lambda t p)^n}{n!} \cdot e^{\lambda t q}$$

$$= \frac{e^{-\lambda t (1-q)} \cdot (\lambda t p)^n}{n!}$$

$$= \frac{e^{-\lambda p t} (\lambda t p)^n}{n!}$$

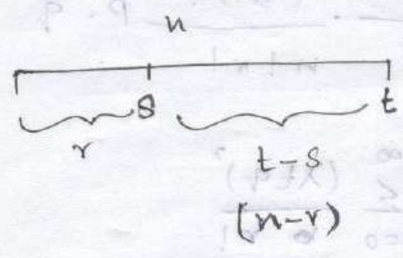
$N(t)$ follows poisson process with parameter λp.

If $X(t)$ is a poisson process, prove that,

$$P[X(s)=r / X(t)=n] = nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r}, \quad s<t$$

Proof:.

$$P[X(s)=r \mid X(t)=n] = \frac{P[X(s)=r \cap X(t)=n]}{P[X(t)=n]}$$



$$= \frac{P\left[X(s)=r \cap X(t-s)=n-r\right]}{P\left[X(t)=n\right]}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^{r}}{r!}\cdot\dfrac{e^{-\lambda(t-s)}\left(\lambda(t-s)\right)^{n-r}}{(n-r)!}}{\dfrac{e^{-\lambda t}(\lambda t)^{n}}{n!}}$$

$$= \frac{e^{-\lambda s}(\cancel{\lambda} s)^{r}}{r!}\cdot\frac{e^{-\cancel{\lambda t}}\,e^{\lambda s}\,\cancel{\lambda}^{n}\,\cancel{\lambda}^{-r}(t-s)^{n-r}}{n-r!}\times\frac{n!}{e^{-\lambda t}\lambda^{n}t^{n}}$$

$$= \frac{nC_r\, s^{r}\, t^{n-r}\left(1-\frac{s}{t}\right)^{n-r}}{t^{n}}$$

$$= \frac{nC_r\, s^{r}\, \cancel{t^{n}}\, t^{-r}\left(1-\frac{s}{t}\right)^{n-r}}{\cancel{t^{n}}}$$

$$= nC_r\left(\frac{s}{t}\right)^{r}\left(1-\frac{s}{t}\right)^{n-r}$$

$$P\left[X(s)=r \,/\, X(t)=n\right] = nC_r\left(\frac{s}{t}\right)^{r}\left(1-\frac{s}{t}\right)^{n-r}$$

If customers arrive at a counter in accordance with the poisson process with a mean rate of 3 per minutes. Find the probability that the interval b/w two consecutive arrivals is

(a) More than 1 minute

(b) b/w 1 minute and 2 minute.

(c) 4 minute (or) less.

Solution:

$$P[T>1] = \int_{-\infty}^{\infty} \lambda e^{-\lambda t} \, dt$$

$$= \int_{1}^{\infty} 3 e^{-3t} \, dt$$

$$= 3 \left[\frac{e^{-3t}}{-3} \right]_{1}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-3} \right]$$

$$= -\left[0 - 0.4978 \right]$$

$$= 0.4978$$

$$P[1<T<2] = \int_{1}^{2} \lambda e^{-\lambda t} \, dt$$

$$= \int_{1}^{2} 3 e^{-3t} \, dt$$

$$= 3 \left[\frac{e^{-3t}}{} \right]^{2} dt$$

$$= -\left[e^{-6} - e^{-3}\right]$$

$$= -\left[0.002 - 0.0497\right]$$

$$= -\left[-0.0477\right]$$

$$= 0.0477$$

$$P[T \leq 4] = \int_0^4 3e^{-3t}\,dt$$

$$= 3\left[\frac{e^{-3t}}{-3}\right]_0^4$$

$$= -\left[e^{-12} - e^{0}\right]$$

$$= -\left[0.00000614 - 1\right]$$

$$= -\left[-0.999\right]$$

$$= 0.999$$

Gaussian (Normal) process:

A real value random process $X(t)$ is said to be a Gaussian process (or) a normal process, if the random variables $X(t_1), X(t_2) \cdots X(t_n)$ are jointly normal for a free $n = 1, 2 \ldots$ and for any set of t_i's

Let $X(t)$ is a gaussian random process with $\mu[X(t)] = 10$ and $C_{XX}(t_1, t_2) = 16e^{-|t_1 - t_2|}$.

Find (i) $X(10) \leq 8$

(ii) $|X(10) - X(6)| \leq 4$

Solution:.

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E[X(t_1) \cdot X(t_2)]$$

$$C_{XX}(t_1,t_1) = R_{XX}(t_1,t_1) - E[X(t_1)]^2$$

$$= E[X^2(t_1)] - E[X(t_1)]^2$$

$$C_{XX}(t_1,t_1) = Var(X(t_1)) \quad \text{---} (1)$$

Given:

$$C_{XX}(t_1,t_2) = 16e^{-|t_1 - t_2|}$$

Let $t = t_1 = t_2$.

$$\cancel{C_{XX}(t_1,t_1)}$$

$$C_{XX}(t,t) = 16e^{-|t-t|}$$

$$= 16e^0$$

$$C_{XX}(t,t) = 16 \quad \text{---} (2)$$

Sub (2) in (1)

$$Var[X(t)] = C_{XX}(t,t) = 16$$

$\{X(t)\}$ is a random Variable with Mean 10 and Variance 16.

(i) $P[X(10) \leq 8]$

Let $Z = \frac{X-\mu}{\sigma}$

$$Z = \frac{X-10}{\sqrt{16}}$$

$$Z = \frac{X-10}{4}$$

$$P[X(10) \leq 8) = P\left[Z \leq \frac{8-10}{4}\right)$$

$$= P[Z \leq 0.5]$$

$$= 0.5 - P(0 < Z < 0.5)$$

$$= 0.5 - 0.1915$$

$$= 0.3085$$

(ii) $P[|X(10) - X(6)| \leq 4]$

Let $U = X(10) - X(6)$.

$$E[U] = E[X(10)] - E[X(6)]$$

$E[X(t)] = \text{Mean} = 10$

$$= 10 - 10$$

$$E[U] = 0$$

$$E[U^2] = \cancel{E[X(10)] - E[X(6)]}$$

$$= E[[X(10) - X(6)]^2]$$

$$= E[X^2(10) + X^2(6) - 2X(10)X(6)]$$

$$= E[X^2(10)] + E[X^2(6)] - 2E[X(10)]E[X(6)]$$

$$= E[X^2(10)] + E[X^2(6) - \cancel{2E[X(10)]} 2$$

$$Var(U) = E[U^2] - [E[U]]^2$$

$$= E[X^2(10)] + E[X^2(6)] - 2\,cov\,(10,6).$$

$$= cov\,(10,10) + cov\,(6,6) - 2\,cov(10,6).$$

$$= 16 + 16 - 32\,e^{-4}$$

$$= 32 - 32\,(0.0183)$$

$$= 32 - 0.5861$$

$$= 31.413$$

$$\sigma_U^2 = 31.413$$

$$\boxed{\sigma_U = 5.604}$$

$$P[|X(10) - X(6)| \le 4] = P\{|U| \le 4\}$$

$$= P[-4 \le U \le 4]$$

$$Z = \frac{U - \mu}{\sigma} = \frac{U - 0}{5.604}$$

$U = -4$

$$Z = \frac{-4-0}{5.604} = -0.7136$$

$U = 4$

$$Z = \frac{4-0}{5.604} = 0.7136$$

$$P[-4 \le U \le 4] = P[-0.7136 < Z < 0.7136]$$

$$= P\cancel{[0.7136 < 0] +}$$

$= P[0 < z < 0.7136] + P[0 < z < 0.7136]$

$= 2P[0 < z < 0.7136]$

$= 2(0.611)$

$= 0.5222$

Suppose $X(t)$ is a normal process with mean $\mu(t) = 3$ and $C(t_1,t_2) = 4e^{-0.2|t_1-t_2|}$.

Find (i) $X(5) \le 2$

(ii) $|X(8) - X(5)| \le 1$

Solution:-

$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - E[X(t_1)X(t_2)]$

$C_{XX}(t_1, t_1) = R_{XX}(t_1, t_1) - E[X(t_1)]^2$

$= E[X^2(t_1)] - E[X(t_1)]^2$

$C_{XX}(t_1, t_1) = Var(X(t_1))$ —— (1)

Given

$C_{XX}(t_1, t_2) = 4e^{-0.2(t_1 - t_2)}$

Put $t = t_1 = t_2$

$C_{XX}(t, t) = 4e^{-0.2(0)}$

$= 4$

$C_{XX}(t, t) = 4$ —— (2)

substitute (2) in (1)

$\ldots] = C_{XX}(t, t) = 4$

mean 3 and Variance 4.

(i) To Find $P[x(5)] \leq 2$;

Let $z = \frac{x-\mu}{\sigma}$

$z = \frac{x-3}{\sqrt{4}}$

$z = \frac{x-3}{2}$

$P[x(5)] \leq 2 = P\left[z \leq \frac{2-3}{2}\right]$

$= P[z \leq -0.5]$

$= 0.5 - P[0 < z < 0.5]$

$= 0.5 - 0.1915$

$P[x(5)] \leq 2 = 0.3085$

(ii) $P[x(8) - x(5)] \leq 1$

Let $U = x(8) - x(5)$

$E[U] = E[x(8)] - E[x(5)]$

$= 3 - 3$

$= 0$

$E[U] = 0.$

$E[U^2] = E[[x(8) - x(5)]^2]$

$= E[x^2(8) + x^2(5) - 2x(8)x(5)]$

$= E[x^2(8)] + E[x^2(5)] - 2E[x(8)]E[x(5)]$

$= E[x^2(8)]$

$= E[X^2(8)] + E[X^2(5)] - 2\,cov(8,5)$

$= cov(8,8) + cov(5,5) - 2\,cov(8,5)$

$= 4 + 4 - 8e^{-0.6}$

$= 8 - 8e^{-0.6}$

$= 8(1 - e^{-0.6})$

$\sigma_U^2 = 3.609$

$\sigma_U = 1.899$

$P[|X(8) - X(5)| \leq 1] = P[|U| \leq 1]$

$= P[-1 < U < 1]$

Let $z = \frac{U - \mu}{\sigma} = \frac{U - 0}{1.899}$

Put $U = -1 \Rightarrow z = \frac{-1}{1.899} = -0.5265$

Put $U = 1 \Rightarrow z = \frac{1}{1.899} = 0.5265$

$P[|X(8) - X(5)| \leq 1] = P[-0.5265 < z < 0.5265]$

$= 2P[0 < z < 0.5625]$

$= 2(0.2019)$

$= 0.4038$

$P[|X(8) - X(5)| \leq 1] = 0.4038$

Random Telegram process:

Random Telegram process is a discrete random process $x(t)$, satisfies the following conditions

(i) $x(t)$ assumes only two values -1 and 1

(ii) $P[X(0)=1] = 1/2 = P[X(0)=-1]$

(iii) The no. of level transistions (or) flips ~~[illegible]~~ $N(t)$ in the interval length t follows poisson process

$$P[N(t)=r] = \frac{e^{-\lambda t}\lambda^{r}}{r!}, \quad r = 1, 2, 3 \ldots$$

Sine wave Process:

A sine wave random process is represented as $x(t) = A \sin(\omega t + \theta)$, where Amplitude A (or) Frequency ω (or) phase (or) any combination of these three may be removed.

For the sine wave process $X(t) = Y \cos \omega_0 t$, $-\infty < t < \infty$, ω_0 = Constant. The amplitude Y: is a random Variable with uniform distribution in the interval 0 to 1. Check whether the process is stationary or not.

Solution!

Given Y is uniformly distributed in the interval $(0,1)$

$$F(y) = \frac{1}{1-0} = 1$$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t) \cdot F(y)\, dy$$

$$= \int_0^1 y \cos w_0 t \; (1)\, dy$$

$$= \cos w_0 t \int_0^1 y\, dy$$

$$= \cos w_0 t \left[\frac{y^2}{2}\right]_0^1$$

$$= \cos w_0 t \left(\tfrac{1}{2} - 0\right)$$

$$= \tfrac{1}{2} \cos w_0 t$$

Since the mean is time dependent

Thus the process is not stationary.