UNIT-II

TWO DIMENSIONAL RANDOM VARIABLE

Let 'S' be the Sample Space. Let $X = X(s)$ and $Y = Y(s)$ be the two functions each assigning a real number to each outcome $s \in S$. Then (X, Y) is a two dimensional random variable.

Note:

The two random variables of (X, Y) are said to be independent if,

$$P[X = x_i / Y = y_j] = P[X = x_i]\ P(Y = y_j)$$

$$P_{ij} = P_{i.} \times P_{.j}$$

Problems based on marginal distribution

1. From the following joint distribution of X and Y. Find the marginal distribution.

Y \ X	0	1	2
0	3/28	9/28	3/28

The marginal distribution of X are

$$P[X=0] = P(0,0) + P(0,1) + P(0,2)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{1}{28}$$

$$= \frac{10}{28}$$

$$P[X=1] = P(1,0) + P(1,1) + P(1,2)$$

$$= \frac{9}{28} + \frac{3}{14} + 0$$

$$= \frac{15}{28}$$

$$P[X=2] = P(2,0) + P(2,1) + P(2,2)$$

$$= \frac{3}{28}$$

The marginal distribution of Y are

$$P[Y=0] = P[0,0] + P[1,0] + P[2,0]$$

$$= \frac{3}{28} + \frac{9}{28} + \frac{3}{28}$$

$$= \frac{15}{28}$$

$$P[Y=1] = P(0,1) + P(1,1) + P(2,1)$$

$$= \frac{3}{14} + \frac{3}{14}$$

$$= \frac{6}{14}$$

$$P[Y=2] = P(0,2) + P(1,2) + P(2,2)$$

$$= \frac{1}{28}$$

The marginal distribution of X & Y are.

Y \ X	0	1	2	P(Y=y)
0	3/28	9/28	3/28	15/28
1	3/14	3/14	0	6/14
2	1/28	0	0	1/28
P(X=x)	5/14	15/28	3/28.	1

1/2/13 2. If the joint p.d.f of (X, Y) is given by,

$P(x,y) = K(2x+3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find the marginal distribution. Also find the probability distribution of $(X+Y)$

Solution:

$P(x,y) = K(2x+3y)$

$P(0,1) = K(2(0)+3(1)) = 3K$.

$P(0,2) = K(2(0)+3(2)) = 6K$.

$P(0,3) = K(2(0)+3(3)) = 9K$.

$P(1,1) = K(2(1)+3(1)) = 5K$

$P(1,2) = K(2(1)+3(2)) = 8K$

$P(1,3) = K(2(1)+3(3)) = 11K$

$P(2,1) = K(2(2)+3(1)) = 7k$

$P(2,2) = K(2(2)+3(2)) = 10k.$

$P(2,3) = K(2(2)+3(3)) = 13k$

To find k :

$y \backslash x$	0	1	2	$P(Y=y)$
1	3k	5k	7k	15k
2	6k	8k	10k	24k
3	9k	11k	13k	33k
$P(X=x)$	18k	24k	30k	72k

$72k = 1$

$$\boxed{K = \frac{1}{72}}$$

The marginal distribution of X & Y is

$Y \backslash X$	0	1	2	$P(Y=y)$
1	$\frac{3}{72}$	$\frac{5}{72}$	$\frac{7}{72}$	$\frac{15}{72}$
2	$6/72$	$8/72$	$10/72$	$24/72$
3	$9/72$	$11/72$	$13/72$	$33/72$
$P(X=x)$	$18/72$	$24/72$	$\frac{30}{72}$	$\frac{72}{72} = 1$

Probability distribution of X+Y

X+Y	Probability
1	$P(0,1) = 3/72$
2	$P(0,2)+P(1,1) = 6/72 + 5/72 = \frac{11}{72}$
3	$P(2,1)+P(1,2)+P(0,3) = \frac{7}{72}+\frac{8}{72}+\frac{9}{72} = \frac{24}{72}$
4	$P(2,2)+P(1,3) = \frac{10}{72}+\frac{11}{72} = \frac{21}{72}$
5	$P(2,3) = \frac{13}{72}$

Problems based on Conditional distribution

3. From the following table for by Variate distribution (X,Y). Find

(i) $P(X \leq 1)$

(ii) $P(Y \leq 3)$

(iii) $P(X \leq 1, Y \leq 3)$

(iv) $P(X \leq 1 / Y \leq 3)$

(v) $P(Y \leq 3 / X \leq 1)$

(vi) $P(X+Y \leq 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution:

X \ Y	6	1	2	3	4	5	P(X=x)
	P(0,6)	P(0,1)	P(0,2)	P(0,3)	P(0,4)	P(0,5)	
0	$3/32$ P(1,6)	0 P(1,1)	0 P(1,2)	$1/32$ P(1,3)	$2/32$ P(1,4)	$2/32$ P(1,5)	$8/32$ P(X=0)
1	$1/8$ P(2,6)	$1/16$ P(2,1)	$1/16$ P(2,2)	$1/8$ P(2,3)	$1/8$ P(2,4)	$1/8$ P(2,5)	$10/16$ P(X=1)
2	$2/64$ P(Y=6)	$1/32$ P(Y=1)	$1/32$ P(Y=2)	$1/64$ P(Y=3)	$1/64$ P(Y=4)	0 P(Y=5)	$8/64$ P(X=2)
P(Y=y)	$\frac{16}{64}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	1

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{28}{32} = \frac{7}{8}$$

(ii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \leq 1, Y \leq 3) = P(0,1) + P(0,2) + P(0,3) +$

$$+ P(1,1) + P(1,2) + P(1,3)$$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{9}{32}$$

(iv) $P(X \le 1 \mid Y \le 3) = \dfrac{P((X \le 1) \cap \cancel{P}(Y \le 3))}{P(Y \le 3)}$

$= \dfrac{9/32}{\frac{23}{64}}$

$= \dfrac{18}{23}$

(v) $P(Y \le 3 \mid X \le 1) = \dfrac{P(Y \le 3 \cap X \le 1)}{P(X \le 1)}$

$= \dfrac{9/32}{7/8}$

$= 9/28.$

(vi) $P(X+Y \le 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4)$

$+ P(1,1) + P(1,2) + P(1,3) +$

$P(2,1) + P(2,2)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$+ \frac{1}{32} + \frac{1}{32}$

$= \frac{13}{32}$

4. The joint probability mass function of X and Y is

$Y \backslash X$	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Find the marginal distribution function of X & Y. Also $P(X \leq 1, Y \leq 1)$ and check if X and Y are independent.

Solution: MDF of X & Y

$Y \backslash X$	0	1	2	$P(Y=y)$
0	0.10	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.30	0.5
$P(X=x)$	0.24	0.38	0.32	1

$$P(X \leq 1, Y \leq 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$$

$$= 0.10 + 0.04 + 0.08 + 0.20$$

$$= 0.42$$

To check X & Y are independent

$$P(X=0) \cdot P(Y=0) = (0.16)(0.24)$$

$$= 0.038$$

$$\neq 0.10$$

8. **Joint Probability distribution function for continuous random Variable.**

$$F[x,y] = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

Marginal distribution functions:

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx.$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

Joint Probability density function:

$$f_{XY}(x,y) = \frac{\partial^2 F[x,y]}{\partial x\,\partial y}.$$

Marginal Probability density function.

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

Conditional probability density function.

$$f(y/x) = \frac{f(x,y)}{f(x)}, \quad f(x) > 0$$

$$f(x/y) = \frac{f(x,y)}{f(y)}, \quad f(y) > 0$$

5) Show that the function ~~f(x,y) =~~

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0<x<1,\ 0<y<1 \\ 0, & \text{Otherwise.} \end{cases}$$

Find JDF of X & Y.

Solution:

(i) $f(x,y) \geq 0$ in $0 \leq x, y \leq 1$

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy.$$

$$= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\, dx\, dy.$$

$$= \frac{2}{5}\int_0^1 \left[\cancel{\frac{2x^2}{2}\Big]_0^1 dy}\right. \left[\frac{2x^2}{2} + 3xy\right]_0^1 dy$$

$$= \frac{2}{5}\int_0^1 1+3y\, dy$$

$$= \frac{2}{5}\left[y + \frac{3y^2}{2}\right]_0^1$$

$$= \frac{2}{5}\left(\left[1+\frac{3}{2}\right] - 0\right)$$

$$= \frac{2}{5}\left(\frac{5}{2}\right)$$

6. The Joint p.d.f of random Variable X and Y is given by

$$f(x,y) = Kxy\, e^{-(x^2+y^2)}, \quad x>0,\ y>0$$

Find the Value of k. and prove also that X and Y are independent.

Solution:

$$F[x,y] = Kxy\, e^{-(x^2+y^2)}$$

$$\int_0^\infty \int_0^\infty Kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

$$K \int_0^\infty x\, e^{-(x^2)}\, dx \int_0^\infty y\, e^{-y^2}\, dy = 1$$

$$K \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

To prove X and Y independent:

(ie) $f(x) \cdot f(y) = f(x,y)$

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} [f(x,y)]\, dy$$

$$= \int^{\infty} Kxy\, e^{-(x^2+y^2)}\, dy$$

$$= 4 x e^{-x^2} \int_0^\infty y e^{-y^2} \, dy$$

$$= \overset{2}{\cancel{4}} x e^{-x^2} \cdot \frac{1}{2}$$

$$f(x) = 2 x e^{-x^2}$$

$$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx$$

$$= \int_0^\infty K x y \, e^{-(x^2+y^2)} \, dx$$

$$= 4 y e^{-y^2} \int_0^\infty x e^{-x^2} \, dx$$

$$= 4 y e^{-y^2} \cdot \frac{1}{2}$$

$$f(y) = 2 y e^{-y^2}$$

$$f(x) \cdot f(y) = 2 x e^{-x^2} \cdot 2 y e^{-y^2}$$

$$= 4 x y e^{-(x^2+y^2)}$$

$$= f(x,y)$$

X & Y are independent

7. Let X and Y have JDF,

$f(x,y) = 2 \quad 0 < x < y < 1$. Find the MDF. Find the CDF of $(Y/X=x)$

Solution :-

M.D.f of X,

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_x^1 2\, dy.$$

$$= 2\,[y]_x^1$$

$$f(x) = 2\,[1-x]$$

M.d.f of Y,

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^y 2\, dx$$

$$= 2\,[x]_0^y$$

$$= 2y$$

The c.d.f of y given $X=x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{2}{2(1-x)}$$

$$= \frac{1}{}$$

8. The jdf of the random Variable X and Y is given by

$$f(x,y) = \begin{cases} 8xy, & 0<x<1, \\ & 0<y<x \\ 0, & \text{otherwise.} \end{cases}$$

find (i) $f_X(x)$

(ii) $f_Y(y)$

(iii) $f(Y/x)$

(i) $f_X(x) = f(x) = \int_0^x 8xy\, dy$

$= 8x\int_0^x y\, dy$

$= 8x\left[\frac{y^2}{2}\right]_0^x$

$= 8x \cdot \frac{x^2}{2}$

$= 4x^3$

$f(x) = 4x^3$

(ii) $f_Y(y) = f(y) = \int_0^1 8xy\, dx$

$= 8y\left[\frac{x^2}{2}\right]_0^1$

$= 8y \cdot \frac{1}{2}$

$= 4y.$

$f(y) = 4y.$

(iii) $f(y/x) = \dfrac{f(x,y)}{f(x)}$

$\cancel{f(x,y)} = \dfrac{8xy}{4y} = \dfrac{8xy}{4x^{2}}$

$f(y/x) = \dfrac{2y}{x^2}$

9. The Joint p.d.f of a two dimensional random Variable (X,Y) is given by.

$$f(x,y) = xy^2 + x^2/8, \quad 0 \le x \le 2, \quad 0 \le y \le 1$$

Compute (i) $P(X>1 \mid Y<1/2)$

(ii) $P(Y<1/2 \mid X>1)$

(iii) $P(X<Y)$

(iv) $P(X+Y \le 1)$

Solution :

(i) $P(X>1/Y<1/2) = \dfrac{P(X>1, Y<1/2)}{P(Y<1/2)}$

$$P(X>1, Y<1/2) = \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\, dx$$

$$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2 y}{8}\right]_0^{1/2} dx$$

$$= \int_1^2 \left[\left[\frac{x}{24} + \frac{x^2}{16}\right] - (0+0)\right] dx$$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48}\right]_1^2$$

$$= \frac{1}{48}\left[(4+8) - (1+1)\right]$$

$$= \frac{1}{48}\left[12-1\right]$$

$$= \frac{10}{48} = \frac{5}{24}$$

$$P(Y < 1/2) = \int_0^{1/2}\int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \left[\cancel{\frac{xy^3}{3}} + \cancel{\frac{x^2y}{8}}\right]_0^{1/2}$$

$$= \cancel{\frac{x}{24}} + \cancel{\frac{x^2}{16}}$$

$$= \int_0^{1/2} \left[\frac{x^2y^2}{2} + \frac{x^3}{24}\right]_0^2 dy$$

$$= \int_0^{1/2} \left(\left[\frac{4y^2}{2} + \frac{8}{24}\right] - [0+0]\right) dy$$

$$= \int_0^{1/2} \left[\frac{4y^2}{2} + \frac{1}{3}\right] dy$$

$$= \left[\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{2}{24} + \frac{1}{6}\right]$$

$$= \frac{6}{24} = \frac{1}{4}$$

$$P\left[X>1 / Y<1/2\right] = \frac{P(X>1,\ Y<1/2)}{P(Y<1/2)}$$

$$= \frac{5/24}{1/4}$$

$$= 5/6$$

$$\therefore P\left[X>1 / Y<1/2\right] = 5/6.$$

(iii) ~~P(X>1 / Y<1/2)~~

$$P(Y<1/2 / X>1) = \frac{P(Y<1/2 ;\ X>1)}{P(X>1)}$$

$$P(X>1) = \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2}y^2 + \frac{x^3}{24}\right)_1^2 dy.$$

$$= \int_0^1 \left[\left[\frac{4y^2}{2} + \frac{8}{24}\right] - \left[\frac{y^2}{2} + \frac{1}{24}\right)\right] dy$$

$$\int_0^1 \left[\left(2y^2 \ldots\right.\right.$$

$$= \left[\frac{2y^3}{3} + \frac{y}{3} - \frac{y^3}{6} - \frac{y}{24}\right]_0^1$$

$$= \left[\frac{2}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{3}{3} - \frac{1}{6} - \frac{1}{24}\right]$$

$$= \left[\frac{24-4-1}{24}\right]$$

$$= \frac{19}{24}$$

$$P(Y<1/2 / X>1) = \frac{P(Y<1/2, X>1)}{P(X>1)}$$

$$= \frac{5/24}{19/24}$$

$$= \frac{5}{19}$$

$$\therefore P[Y<1/2 / X>1] = \frac{5}{19}$$

(iii) $$P(X<Y) = \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy$$

$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left[y^5 + \frac{y^4}{\ } \right]^1$$

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Covariance:

If X and Y are random variables, then covariance between X and Y is defined as

$$Cov(X,Y) = E[XY] - E[X].E[Y].$$

Note:

If X and Y are independent then, $E[XY] = E[X].E[Y]$

$\Rightarrow Cov(X,Y) = 0$

(1) $Cov(aX, bY) = ab\, Cov(X,Y)$

(2) $Cov(X+a, Y+b) = Cov(X,Y)$

(3) $Cov(aX+b, cY+d) = ac\, Cov(X,Y)$

(4) $Var(X_1+X_2) = Var\, X_1 + Var\, X_2 + 2\, Cov(X_1, X_2)$

(5) $Var(X_1-X_2) = Var\, X_1 + Var\, X_2 - 2\, Cov(X_1, X_2)$

(6) If X_1 & X_2 are independent, then

$$Var(X_1 \pm X_2) = Var\, X_1 \pm Var\, X_2.$$

Correlation:

(r) Karl-pearsons Coefficient of correlation:

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_x\, \sigma_y}$$

Where,

$$\sigma_x = \sqrt{\frac{1}{n}\left\{\sum x^2 - \bar{x}^2\right\}}, \quad \bar{x} = \frac{\sum x}{n}$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - \bar{y}^2}, \quad \bar{y} = \frac{\sum y}{n}$$

1) Correlation Coefficient may also be denoted by $\rho(x,y)$ (or) ρ_{xy}

2) If $\rho(x,y) = 0$, we say that x and y are uncorrelated.

3) When $r = 1$, the correlation is perfect and positive.

Two independent variables are uncorrelated. Since $Cov(x,y) = 0$ when X and Y are independent

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} = 0$$

10. Calculate the Correlation Coefficient for the following ~~types~~ height (in inches) of Fathers (x) and their Sons (y)

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

Solution:.

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\Sigma x = 544$	$\Sigma y = 552$	$\Sigma xy = 37560$	$\Sigma x^2 = 37028$	$\Sigma y^2 = 38132$

$$\bar{X} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$$

$$\bar{Y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$$

$$\overline{\Sigma x}^2 = \frac{\Sigma x^2}{n} = \frac{37028}{8} =$$

$$\bar{x}\,\bar{y} = 68 \times 69 = 4692$$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{x}^2}$$

$$= \sqrt{\frac{1}{8}(37028) - (68)^2}$$

$$= 2.121$$

$$\sigma_Y = \sqrt{\frac{1}{n}\Sigma Y^2 - \bar{Y}^2}$$

$$= \sqrt{\frac{1}{8}(38132) - 69^2}$$

$$= 2.345$$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{n}\Sigma XY - \bar{X}\bar{Y}}{\sigma_X \sigma_Y}$$

$$= \frac{\frac{1}{8}(37560) - 4692}{2.121 \times 2.345}$$

$$= 0.6030.$$

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2. Find the correlation coefficient for the following data.

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution:

X	Y	$U = \frac{X-22}{4}$	$V = \frac{Y-24}{6}$	UV	U^2	V^2
10	18	−3	−1	3	9	1
14	12	−2	−2	4	4	4
18	24	−1	0	0	1	0
22	6	0	−3	0	0	9
26	30	1	1	1	1	1
30	36	2	2	4	4	4

$$\bar{u} = \frac{\Sigma u}{n} = \frac{-3}{6} = -0.5$$

$$\bar{v} = \frac{\Sigma v}{n} = \frac{-3}{6} = -0.5$$

$$\bar{U}\bar{V} = -0.5 \times -0.5$$

$$= 0.25$$

$$\sigma_u = \sqrt{\frac{1}{6}\Sigma u^2 - (\bar{u})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$\sigma_v = \sqrt{\frac{1}{6}\Sigma v^2 - (\bar{v})^2}$$

$$= \sqrt{\frac{1}{6}(19) - (0.25)}$$

$$= 1.70$$

$$r(x,y) =$$

$$r(u,v) = \frac{Cov(u,v)}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{n}\Sigma uv - \bar{u}\bar{v}}{\sigma_u \sigma_v}$$

$$= \frac{\frac{1}{6}(12) - 0.25}{(1.70)(1.70)}$$

$$= \frac{1.75}{2.917266} = 0.6$$

3. The Joint probability mass function X and Y is

$X \backslash Y$	-1	1
0	1/8	3/8
1	2/8	2/8

Find the correlation coefficient of X and Y.

Solution:

$X \backslash Y$	-1	1	$P(X=x)$
0	$\frac{1}{8}$ $P(0,-1)$	$\frac{3}{8}$ $P(0,1)$	$\frac{4}{8}$ $P(X=0)$
1	$\frac{2}{8}$ $P(1,-1)$	$\frac{2}{8}$ $P(1,1)$	$\frac{4}{8}$ $P(X=1)$
$P(Y=y)$	$\frac{3}{8}$ $P(Y=-1)$	$\frac{5}{8}$ $P(Y=1)$	1

$$E[X] = \sum x \; P(x)$$

$$= 0 \times \frac{1}{2} + 1 \times \frac{1}{2}$$

$$= \frac{1}{2}$$

$$E[Y] = \sum y \; P(y)$$

$$= -1 \times \frac{3}{8} + 1 \times \frac{5}{8}$$

$$= \frac{1}{4}$$

$$E[X^2] = \sum x^2 \; P(x)$$

$$E[Y^2] = \sum y^2 P(y)$$

$$= (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8}$$

$$= 1.$$

$$E[XY] = \sum xy\, P(x,y)$$

$$= 0 \times -1 \times \frac{1}{8} + 0 \times 1 \times \frac{3}{8} + 1 \times -1 \times \frac{2}{8} + 1 \times 1 \times \frac{2}{8}$$

$$= 0$$

$$\sigma_x^2 = E[X^2] - [E(X)]^2$$

$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}$$

$$\sigma_Y^2 = E[Y^2] - [E(Y)]^2$$

$$= 1 - \left(\frac{1}{4}\right)^2$$

$$= \frac{15}{16}$$

$$\rho(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{E[XY] - E[X]E[Y]}{\sigma_x \sigma_y}$$

$$= \frac{0 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\sqrt{\frac{1}{4}}\sqrt{\frac{15}{16}}}$$

4) If the independent random Variable X and Y have the Variance 36 and 16 respectively. Find the correlation Coefficients b/w X+Y and X-Y.

Solution:

Given:

$Var(X) = 36$

$Var(Y) = 16$

X+Y are independent.

$E[XY] = E[X]E[Y]$

Let $u = X+Y$

$v = X-Y$

$Var(u) = Var(X+Y)$

$= 1^2 Var(X) + 1^2 Var(Y)$

$= 1\times 36 + 1\times 16 = 36+16$

$= 52.$

$\sigma_u^2 = 52$

$Var(v) = Var(X-Y)$

$= 1^2 Var(X) + (-1)^2 Var(Y)$

$= 1\times 36 + 1\times 16$

$= 36+16$

$= 52$

$\sigma_v^2 = 52$

$\sigma_u = \sqrt{52}, \quad \sigma_v = \sqrt{52}$

$Cov(u,v) = E[uv] - E[u]E[v]$

$E[uv] = E[(x+y)(x-y)]$

$= E[x^2 - y^2]$

$E[uv] = E[x^2] - E[y^2]$

$E[u] = E[x+y]$

$= E[x] + E[y]$

$E[v] = E[x-y]$

$= E[x] - E[y]$

$Cov(u,v) = E[x^2] - E[y^2] - [[E[x]]^2 - [[E(y)]]^2]$

$= E[x^2] - E[y^2] - [(E[x])^2] + [(E(y))^2]$

$= [E[x^2] - [(E[x])^2] -$

$[E[y^2] - [E(y)]^2]$

$= Var(x) - Var(y)$

$= 36 - 16$

$= 20.$

$$\rho(u,v) = \frac{Cov(u,v)}{\sigma_u . \sigma_v}$$

$$= \frac{20}{\sqrt{52}.\sqrt{52}}$$

$$= \frac{20}{52}$$

If the joint p.d.f of (X, Y) is given by $f(x,y) = x+y$, $0 \le x, y \le 1$

Find ρ_{xy}.

Solution:

Given

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \; f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x^2y + xy^2 \; dx\, dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= 2/6$$

$$= 1/3$$

$E[X,Y] = 1/3$.

Mdf of X, $f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$= \int_0^1 x + y\,dy$$

$$= \left[xy + \frac{y^2}{2} \right]_0^1$$

$$= xy + \frac{1}{2}$$

$f(x) = x + 1/2$

Marf of Y, $f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^1 (x+y)\, dx$$

$$= \left[\frac{x^2}{2} + xy \right]_0^1$$

$$= \frac{1}{2} + y$$

$f(y) = y + 1/2$

$$E[x] = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

$$= \int_0^1 x\,(x + 1/2)\, dx$$

$$= \int_0^1 x^2 + \frac{x}{2}\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}.$$

$$E[Y] = \int_{-\infty}^{\infty} y \; f(y) \; dy$$

$$= \int_{0}^{1} y \; (y + 1/2) \, dy$$

$$= \int_{0}^{1} (y^2 + y/2) \, dy$$

$$= \left[\frac{y^3}{3} + \frac{y^2}{4} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \; f(x) \; dx$$

$$= \int_{0}^{1} x^2 \, (x + 1/2) \, dx$$

$$= \int_{0}^{1} \left(x^3 + \frac{x^2}{2} \right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6} \right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12$$

$$E[Y^2] = \int^{\infty} y^2 \; f(x) \; d\ldots$$

$$= \int_0^1 \left(y^3 + \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^4}{4} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{10}{24}$$

$$= 5/12.$$

$$Var(X) = E[X^2] - [E(X)]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$\sigma_x^2 = \frac{11}{144} \Rightarrow \sigma_x = \sqrt{\frac{11}{144}}$$

$$Var(Y) = E[Y] - (E(Y))^2$$

$$= \frac{7}{12} - \cancel{\frac{25}{144}}\ \frac{49}{144}$$

$$\sigma_y^2 = \frac{49}{144} \Rightarrow \sigma_Y = \cancel{\sqrt{\frac{59}{144}}}\ \sqrt{\frac{11}{144}}$$

$$P(X,Y) = \frac{Cov(X,Y)}{\sigma_x \ \sigma_y}$$

$$= \frac{E[XY] - E[X]\,E[Y]}{\sigma_x \ \sigma_y}$$

$$= \frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12}$$

$$= \frac{\frac{1}{3} - \frac{49}{144}}{\frac{11}{144}}$$

$$= \frac{48-49}{144} \times \frac{144}{11}$$

$$= \frac{-1}{11}$$

$$\therefore P(x,y) = \frac{-1}{11}$$

If $f(x,y) = \frac{6-x-y}{8}$, $0 \le x \le 2$, $2 \le y \le 4$, find the correlation coefficient between x and y.

Solution:

Given

$$f(x,y) = \frac{6-x-y}{8} \quad \begin{matrix} 0 \le x \le 2 \\ 2 \le y \le 4 \end{matrix}$$

Mdf of x:

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{2}^{4} \frac{6-x-y}{8}\, dy = \frac{1}{8}\left[6y - xy - y^2/2\right]_2^4$$

$$= \left[\frac{6y-xy-y^2}{16}\right]_2^4 = \frac{1}{8}\left[24-4x-16\right]$$

$$= \left[\frac{24-4x-16}{16}\right] - \left[\frac{12-2x-4}{16}\right]$$

$$= \frac{1}{8}\left[\left(24 - 4\overset{8}{x} - \frac{16}{x}\right) - \left(12 - 2y - \overset{2}{4}/x\right)\right]$$

$$= \frac{1}{8}\left[16 - 4x - 10 + 2y\right]$$

$$= \frac{1}{8}\left[6 - 2x\right]$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{0}^{2} \frac{6 - x - y}{8}\, dx$$

$$= \frac{1}{8}\int_{0}^{2} 6 - x - y\, dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[\left[\cancel{72 - \frac{144}{2} - 12y}\right] - [0]\right]$$

$$= \frac{1}{8}\left[\cancel{72 - 72 - 12y}\right]$$

$$= \cancel{\frac{-\overset{3}{12}y}{\overset{2}{8}}} \qquad = \frac{1}{8}\left[12 - \frac{\overset{2}{4}}{2} - 2y\right]$$

$$= \cancel{-3y/2} \qquad = \frac{1}{8}\left[10 - 2y\right]$$

$$= \frac{1}{4}(5 - y)$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

5/6

$$= \int_{2}^{4}\int_{0}^{2} xy \left(\frac{6-x-y}{8}\right) dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\int_{0}^{2} 6xy - x^2y - xy^2\, dx\, dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^2 dy$$

$$= \frac{1}{8}\int_{2}^{4}\left[\frac{6(4)y}{2} - \frac{8y}{3} - \frac{4y^2}{2}\right] dy$$

$$= \frac{1}{8}\int_{2}^{4} 12y - \frac{8y}{3} - 2y^2\, dy$$

$$= \frac{1}{8}\left[\frac{12y^2}{2} - \frac{8y^2}{6} - \frac{2y^3}{3}\right]_2^4$$

$$= \frac{1}{8}\left[\left[\frac{12\times 16}{2} - \frac{8(16)}{6} - \frac{2(64)}{3}\right] - \left[\frac{12\times 4}{2} - \frac{8\times 4}{6} - 2\right.\right.$$

$$= \frac{1}{8}\left[\left[96 - \frac{64}{3} - \frac{128}{3}\right] - \left[24 - \frac{16}{3} - \frac{16}{3}\right]\right]$$

$$= \frac{1}{8}\left[32 - \frac{40}{3}\right]$$

$$= 4 - \frac{5}{3} = \frac{12-5}{3} = \frac{7}{3}$$

$$E[X] = \int_{-\infty}^{\infty} x \; f(x) \; dx$$

$$\cancel{= \int_0^2 x\left(\frac{6-x-y}{8}\right) dx}$$

$$\cancel{= \frac{1}{8}\int_0^2 6x - x^2 - xy \; dx}$$

$$\cancel{= \frac{1}{8}\left[\frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^2}$$

$$\cancel{= \frac{1}{8}\left[\frac{6\times 4}{2} - \frac{8}{3} - \frac{4y}{2}\right]}$$

$$\cancel{= \frac{1}{8}\left[12 - \frac{8}{3} - y\right]}$$

$$= \int_0^2 x\left(\frac{1}{8}(6-2x)\right) dx$$

$$= \frac{1}{8}\int_0^2 6x - 2x^2 \; dx$$

$$= \frac{1}{8}\left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{6(4)}{2} - \frac{2(8)}{3}\right]$$

$$= \frac{1}{8}\left[12 - \frac{16}{3}\right]$$

$$= \frac{1}{8}\left[\frac{36-16}{3}\right]$$

$$= \frac{1}{8}\left[\frac{20}{3}\right]$$

$$= 5/6$$

$$E[X] = 5/6$$

$$E[Y] = \int_{-\infty}^{\infty} y \; f(y) \; dy$$

$$= \int_2^4 y \; \frac{1}{4}(5-y) \; dy$$

$$= \frac{1}{4}\int_2^4 5y - y^2 \; dy$$

$$= \frac{1}{4}\left[\left[\frac{5\times \cancel{16}^{8}}{\cancel{2}} - \frac{64}{3}\right] - \left[\frac{5\times \cancel{4}^{2}}{\cancel{2}} - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left[40 - \frac{64}{3}\right] - \left[10 - \frac{8}{3}\right]\right]$$

$$= \frac{1}{4}\left[\left(\frac{120-64}{3}\right) - \left(\frac{30-8}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{56}{3} - \frac{22}{3}\right]$$

$$= \frac{1}{\cancel{4}_{2}} \times \frac{\cancel{34}^{17}}{3}$$

$$= \frac{17}{6}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \, f(x)\, dx$$

$$= \int_0^2 x^2 \left(\frac{6-2x}{8}\right) dx$$

$$= \frac{1}{8}\int_0^2 6x^2 - 2x^3 \, dx$$

$$= \frac{1}{8}\left[\frac{6x^3}{3} - \frac{2x^4}{4}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{\cancel{6}^{2}\times 8}{\cancel{3}} - \frac{2\times \cancel{16}^{4}}{\cancel{4}}\right]$$

$$= \frac{1}{8}\left[16 - 8\right]$$

$$= 1$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 \, f(y) \, dy$$

$$= \int_2^4 y^2 \left(\frac{5-y}{4}\right) dy$$

$$= \frac{1}{4}\int_2^4 5y^2 - y^3 \, dy$$

$$= \frac{1}{4}\left[\frac{5y^3}{3} - \frac{y^4}{4}\right]_2^4$$

$$= \frac{1}{4}\left\{\left[\frac{5\times 64}{3} - \frac{\cancel{256}^{\,64}}{\cancel{4}}\right] - \left[\frac{5\times 8}{3} - \frac{\cancel{16}^{\,4}}{\cancel{4}}\right]\right\}$$

$$= \frac{1}{4}\left\{\left[\frac{320}{3} - 64\right] - \left[\frac{40}{3} - 4\right]\right\}$$

$$= \frac{1}{4}\left[\frac{280}{3} - 60\right]$$

$$= \frac{1}{4}\left[\frac{280-180}{3}\right]$$

$$= \frac{1}{\cancel{4}}\left(\frac{\cancel{100}^{\,25}}{3}\right)$$

$$= 25/3$$

$$Var(X) = \sigma_X^2 = E[X^2] - (E(X))^2$$

$$= 1 - (5/6)^2$$

$$= 1 - \frac{25}{36}$$

$$= \frac{36-25}{}$$

$$\sigma_x = \sqrt{\frac{11}{36}}$$

$$Var(Y) = \sigma_Y^2 = E[Y^2] - (E(Y))^2$$

$$= \frac{25}{3} - \left(\frac{17}{6}\right)^2$$

$$= \frac{25}{3} - \frac{289}{36}$$

$$= \frac{300-289}{36}$$

$$= \frac{11}{36}$$

$$\sigma_Y = \sqrt{11/36}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{E[XY] - E[X]E[Y]}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{\frac{7}{3} - \frac{5}{6} \cdot \frac{17}{6}}{\sqrt{\frac{11}{36}} \cdot \sqrt{\frac{11}{36}}}$$

$$= \frac{\frac{7}{3} - \frac{85}{36}}{\frac{11}{36}}$$

$$= \frac{84-85}{\cancel{36}} \times \frac{\cancel{36}}{11}$$

Correlation Coefficient:

$$\gamma = \pm\sqrt{b_{xy} \times b_{yx}}$$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

1. From the following data, Find

(i) The two regression eqn.

(ii) The Coefficient of correlation between the marks in economics and Statistics.

(iii) The most likely marks in Statistics, when marks in economics 30.

Marks in economics	25	28	35	32	31	36	29	38	34	3
Marks in Statistics	43	46	49	41	36	32	31	30	33	3

$\cancel{b_{yx} = 0.6643}$ $\boxed{b_{yx} = -0.6643}$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= -\frac{93}{398}$$

$\cancel{① \Rightarrow y} = -0.2337$

$$\boxed{b_{xy} = -0.2337}$$

① $\Rightarrow y - 38 = -0.6643(x-32)$

$$y - 38 = -0.6643\cancel{x} \times (-0.6643x - 32)$$

$$y = -0.6643x \times 21.2576 + 38$$

$$y = +0.6643 \times 59.2576$$

② $\Rightarrow x - 32 = -0.2337(y-38)$

$$x = -0.2337y \times (-0.2337x - 38) + 32$$

$$= -0.2337y \times 8.8806 + 32$$

$$x = 40.8806 - 0.2337y.$$

(ii) Coefficient of Correlation

$$r = \pm\sqrt{b_{yx} \times b_{xy}}$$

$$= \pm\sqrt{(-0.6643)(-0.2337)}$$

$$= \pm\sqrt{0.152}$$

The tangent of the angle between the lines of regression of Y on X and X on Y is 0.6 and $\sigma_x = \frac{1}{2}\sigma_y$. Find the correlation coefficient between X and Y.

$\tan\theta = 0.6 \quad \sigma_x = 0.5\sigma_y$

Solution:

Given:

$\tan\theta = 0.6.$

$\sigma_x = 0.5\,\sigma_y.$

Angle b/w lines of regression is

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{(0.5\sigma_y)\,\sigma_y}{(0.5\sigma_y)^2+\sigma_y^2}\right)$$

$$= \frac{1-r^2}{r}\left(\frac{0.5\,\sigma_y^2}{0.25\sigma_y^2+1\sigma_y^2}\right)$$

$$0.6 = \frac{1-r^2}{r}\left(\frac{0.5\,\cancel{\sigma_y^2}}{1.25\,\cancel{\sigma_y^2}}\right)$$

$$\frac{1-r^2}{r} = \frac{(1.25)(0.6)}{0.5}$$

$$= \frac{0.75}{0.5}$$

$$r^2 + 1.5r - 1 = 0$$

$$r = \frac{1}{2}, -2 \quad (-2 \text{ is not possible})$$

$$\boxed{r = 1/2}$$

19|2|2013 Transformation of two dimensional random Variable:

$$f_{uv} = f_{xy}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$f_v(u,v) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, du$$

If the joint p.d.f of X, Y is given by $f_{XY}(x,y) = x+y$, $0 \le x, y \le 1$. Find the p.d.f of $U = XY$.

Solution:

Step 1:

To find joint p.d.f of X & Y.

Given:

$$f_{XY}(x,y) = x+y.$$

Step 2:

Introdu...

Step 3:

Expressing the above eqn as

$x = g_1(u,v) \qquad y = g_2(u,v)$

$u = xy \qquad v = y$

$u = x \cdot v \qquad y = v$

$x = \frac{u}{v}$

$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial x}{\partial v} = \frac{-u}{v^2}$

$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = 1$

Step 4:

Find $|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0$$

$$= \frac{1}{v}$$

$$|J| = \frac{1}{v}$$

Step 5:

To find pdf of (u,v)

$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$

$$= \left(\frac{u}{v} + v\right) \cdot \frac{1}{v}$$

Step 6:- Changing the domain, (x, y) into domain (u, v)

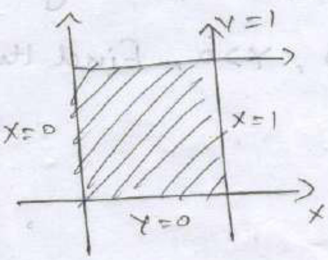
$u = v$

u	0	1	2	3
v	0	1	2	3

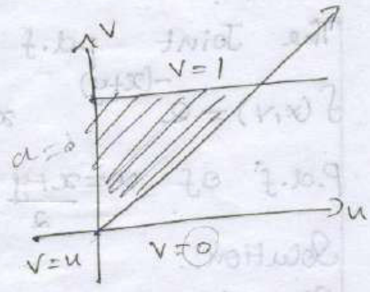
$0 \le y \le 1 \Rightarrow 0 \le v \le 1$

$0 \le x \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$

$\Rightarrow 0 \le u \le v$



(x, y)



(u, v)

P.d.f of (u, v) is given by

$$f_{uv}(u, v) = \frac{1}{v}\left(\frac{u}{v} + v\right) \qquad \begin{array}{l} 0 \le u \le v \\ 0 \le v \le 1 \end{array}$$

Step 7:

To find the p.d.f of $(u = xy)$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u, v)\, dv$$

$$= \int_u^1 \frac{1}{v}\left(\frac{u}{v} + v\right) dv$$

$$= \int_u^1 \frac{u}{v^2} + 1 \; dv$$

$$= \left[u \cdot \frac{v^{-1}}{} + v\right]^1$$

$$= \left[(-u+1) - (-1+u)\right] \quad \begin{matrix} -u+1+1-u \\ 2-2u \\ 2(1-u) \end{matrix}$$

$$= -u+1+1-u$$

$$= 2-2u$$

$$= 2(1-u)$$

$$f_U(u) = 2(1-u) \quad 0 \le u \le 1$$

The Joint pd.f of X & Y is given by $f(x,y) = e^{-(x+y)}$, $x>0$, $y>0$, Find the P.d.f of $u = \frac{x+y}{2}$

Solution:

Step 1:

To find Joint pdf of X, Y.

Given

$$f(x,y) = e^{-(x+y)}, \quad x>0, \ y>0.$$

Step 2:

Introducing new random Variable.

Given, $u = \frac{x+y}{2}$,

Let $v = y$.

Step 3:

Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$u = \frac{x+y}{2}$ $\qquad v = y$

$u = \frac{x+v}{2}$ $\qquad y = v$

$2u = x+v$

$$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial x}{\partial v} = -1$$

$$\frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial v} = 1$$

Step 4 :-

Find $|J| = \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$

$$|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 2$$

Step 5:.

To find pdf of (u,v)

$$f_{uv}(u,v) = f_{xy}(x,y)\,|J|$$

$$= \cancel{(x+y)} \; \frac{e^{-(x+y)}}{2}$$

$$= e^{-(x+y)} \cdot 2 \qquad x>0$$

$$\qquad\qquad 2u-v>0 \qquad y>0$$

$$= 2e^{-2u} \qquad 2u>v \qquad v>0.$$

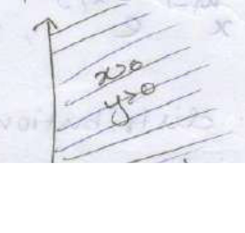
$$u > v/2$$

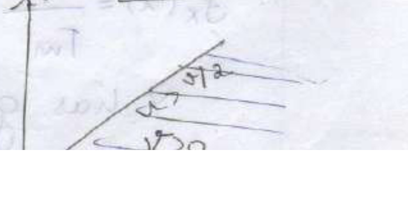
Step 6: Changing the domain (x,y) into (u,v)

$x>0 \Rightarrow u > v/2$

$y>0 \Rightarrow v>0.$

v	0	1	2	3
u	0	1/2	1	1.5





Step T:-

To find the pdf of $u = \frac{x+y}{2}$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$= \int_0^{2u} 2e^{-2u}\, dv$$

$$= 2e^{-2u} \int_0^{2u} dv$$

$$= 2e^{-2u} [v]_0^{2u}$$

$$= 2e^{-2u} \cdot [2u - 0]$$

$$= 4ue^{-2u} \quad u > 0.$$

The random Variable X and Y are Statistically independent having gamma Variable with parameters $(m, 1/2)$ and $(n, 1/2)$ respectively. Derive the p.d.f of a random variable $U = \frac{X}{X+Y}$

Solution:

Step 1: To find Joint p.d.f of X, Y.

X has a gamma distribution with parameters $(m, 1/2)$

$$f_X(x) = \frac{1}{\Gamma m} \frac{1}{2^m} x^{m-1} e^{-x/2}, \quad x > 0.$$

Y has gamma distribution with

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$$= \left[\frac{1}{\Gamma m} \cdot \frac{1}{2^m} x^{m-1} e^{-x/2}\right]\left[\frac{1}{\Gamma n} \frac{1}{2^n} y^{n-1} e^{-y/2}\right]$$

Step 2: Given Introducing new random variable.

$$U = \frac{X}{X+Y}$$

Let $V = X+Y$.

Step 3: Expressing the above eqn as

$x = g_1(u,v)$ & $y = g_2(u,v)$

$$u = \frac{x}{x+y} \qquad v = x+y.$$

$$u = \frac{x}{v} \qquad v = uv + y$$

$$y = v - uv$$

$$x = uv \qquad y = v(1-u)$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1-u.$$

Step 4:

To find $|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix}$$

$$= v(1-u) - (-uv)$$

$$|J| = v$$

Step 5: To find pdf of (u,v)

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$

$$= v\left[\frac{1}{\Gamma m}\cdot\frac{1}{2^m}\cdot x^{m-1}\cdot e^{-x/2}\right]\left[\frac{1}{\Gamma n}\,\frac{1}{2^n}\,y^{n-1}e^{-y/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\,\Gamma n}\cdot\frac{1}{2^{m+n}}\,x^{m-1}\,y^{n-1}\,e^{-\frac{(x+y)}{2}}\right]$$

$$= v\left[\frac{1}{\Gamma m\,\Gamma n}\cdot\frac{1}{2^{m+n}}\,(uv)^{m-1}\,(v(1-u))^{n-1}\cdot e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\,\Gamma n\,2^{m+n}}\,u^{m-1}\,v^{m-1}\,v^{n-1}\,(1-u)^{n-1}\,e^{-v/2}\right]$$

$$= v\left[\frac{1}{\Gamma m\,\Gamma n\,2^{m+n}}\,u^{m-1}\cdot v^{m+n-2}\cdot(1-u)^{n-1}\cdot e^{-v/2}\right]$$

$$f_{UV}(u,v) = \left[\frac{1}{\Gamma m\,\Gamma n\,2^{m+n}}\,u^{m-1}\,v^{m+n-1}\,(1-u)^{n-1}\,e^{-v/2}\right]$$

Step 6:

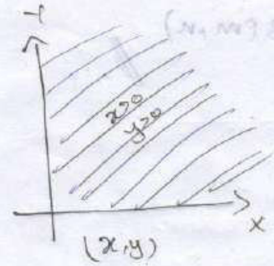
Changing the domain x, y into (u,v)

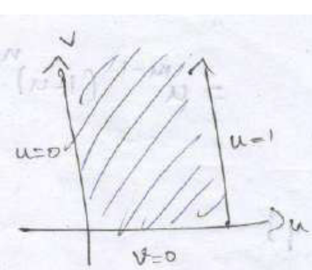
$x>0 \Rightarrow uv>0 \Rightarrow v>0.$

$y>0 \Rightarrow v-uv>0 \Rightarrow v>uv$

$\therefore \Rightarrow 1>u$

$\Rightarrow 0\le u<1$





Step I:

To find the p.d.f of $u = \frac{x}{x+y}$

$$f_u(u) = \int_{-\infty}^{\infty} f_{uv}(u,v)\, dv$$

$$= \int_0^{\infty} \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} v^{m+n-1} e^{-v/2} (1-u)^{n-1} \cdot dv$$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} v^{m+n-1} \cdot e^{-v/2}\, dv$$

Put $v/2 = w \Rightarrow v = 2w$

$dv = 2\, dw$

$v \to 0 \Rightarrow w \to 0$

$v \to \infty \Rightarrow w \to \infty$

$$= \frac{1}{\Gamma m \Gamma n\, 2^{m+n}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} (2w)^{m+n-1}\, 2\, dw$$

$$= \frac{1}{\Gamma m \Gamma n\, \cancel{2^{m+n}}} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} \cdot \cancel{2^{m+n}} \cdot w^{m+n-1}\, dw.$$

$$= \frac{1}{\Gamma m \Gamma n} u^{m-1} (1-u)^{n-1} \int_0^{\infty} e^{-w} w^{(m+n)-1}\, dw.$$