JEPPIAAR INSTITUTE OF TECHNOLOGY

“Self-Belief | Self Discipline | Self Respect”

DEPARTMENT

OF

ELECTRONICS AND COMMUNICATION ENGINEERING

LECTURE NOTES

MA8451-PROBABILITY AND RANDOM PROCESSES

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Unit - I
Random Variables

Probability of an event :

$$P(A) = \frac{\text{favourable cases}}{\text{Possible cases}}.$$

Random experiment :

All outcomes are known, but we can't Predict the exact outcome.

Trial

Performing an experiment.

Sample Space :

All possible outcomes of an experiment

eg 1) Tossing a coin 2) rolling a die

$S = \{H, T\}$ $S = \{1, 2, 3, 4, 5, 6\}$

Event :

Subset of a Sample Space.

eg In the toss of a coin, Let A be the event of getting head.

Equally Likely.

Cannot be expected to happen in Preferrence to any other.

eg Turning up of the head or tail is equally likely.

Mutually Exclusive:

Occurrence of one of them does not Prevent the occurrence of others.

eg Either head or tail will turn up. Both cannot happen at the same time.

Exhaustive Events:

A set is exhaustive if it includes all possible outcomes of a trial.

Axioms of Probability:

Let S be a Sample Space. To each event A, there is a number $p(A)$ associated, Probability of A Satisfying the following conditions

(i) $p(A) \geq 0$

(ii) $P(S) = 1$

(iii) If $A_1, A_2, \ldots A_n$ are mutually exclusive events, then

Addition theorem:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional Probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Multiplication theorem:

$$P(A \cap B) = P(A) \cdot P(B/A)$$

Independent events

$$P(A \cap B) = P(A) \cdot P(B).$$

Random Variables:

It is a function X which assigns a number to every outcome of a random experiment.

eg Tossing two unbiased coins.

Outcomes : HH, HT, TH, TT

Random Variable X : No. of heads.

(Assigning real no's) : (2, 1, 1, 0)

Mathematical defn : $X: S \longrightarrow R$

(1) A random Variable X has the following Probability distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

find (i) the value of k

(ii) $P[1.5 < X < 4.5 / X > 2]$

(iii) the Smallest value of λ for which $P[X \leq \lambda] > \frac{1}{2}$.

Soln

(i) WKT $\sum P_i = 1$

$$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$9k+10k^2=1$$

$$10k^2+9k-1=0$$

$$(k+1)(10k-1)=0$$

$$k=-1 \quad k=\frac{1}{10}$$

(ii) $$P[1.5 < X < 4.5 / X > 2] = \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P[X > 2]}$$

$$= \frac{P[2 < X < 4.5]}{P[X > 2)} = \frac{P[X=3]+P[X=4]}{P[X>2]}$$

$$= \frac{\frac{2}{10}+\frac{3}{10}}{\frac{6}{10}+\frac{10}{100}} = \frac{5}{7}$$

(iii) $P[X \leq \lambda] > \frac{1}{2}$

$\Rightarrow P[X \leq 3] = \frac{1}{2}$

(2) A random variable X takes the values 1, 2, 3 & 4 such that $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]$ find the Probability distribution and cumulative distribution function of X.

Soln

Let

$$2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4] = k$$

$$\left.\begin{aligned} P[X=1] &= k/2 \\ P[X=2] &= k/3 \\ P[X=3] &= k \\ P[X=4] &= k/5 \end{aligned}\right\} \text{———} (1)$$

WKT $\sum p_i = 1$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\frac{61k}{30} = 1$$

$$\boxed{k = \frac{30}{61}}$$

Sub $k = \frac{30}{61}$ in (1)

$$P[X=1] = \frac{30}{61} \times \frac{1}{2} = \frac{15}{61}$$

$$P[X=2] = \frac{30}{61} \times \frac{1}{3} = \frac{10}{61}$$

(i) Probability distribution function is

X	1	2	3	4
$P(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

ii) Cummulative distribution function is

X	$F[x] = P[X \le x]$
1	$\frac{15}{61}$
2	$\frac{15}{61}+\frac{10}{61} = \frac{25}{61}$
3	$\frac{15}{61}+\frac{10}{61}+\frac{30}{61} = \frac{55}{61}$
4	$\frac{15}{61}+\frac{10}{61}+\frac{30}{61}+\frac{6}{61} = \frac{61}{61} = 1$

(3) A. R.V X has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

,

i) Determine a

ii) Evaluate $P(X<3)$, $P(X \ge 4)$, $P(0 < X \le 5)$

iii) Find the distribution function of X

Soln

WKT $\sum P(x) = 1$

x	0	1	2	3	4	5	6	7	8
$P(x)$	$1/81$	$3/81$	$5/81$	$7/81$	$9/81$	$11/81$	$13/81$	$15/81$	$17/81$
$F[x]$	$1/81$	$4/81$	$9/81$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1.

$$P(X<3) = P(0) + P(1) + P(2)$$
$$= \frac{1}{81} + \frac{3}{81} + \frac{5}{81} = \frac{9}{81}$$

$$P(X \geq 4) = P(4) + P(5) + P(6) + P(7) + P(8)$$
$$= \frac{9}{81} + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + \frac{17}{81}$$
$$= \frac{65}{81}$$

$$P(0 < X \leq 5) = P(X = 1, 2, 3, 4, 5)$$
$$= \frac{3}{81} + \frac{5}{81} + \frac{7}{81} + \frac{9}{81} + \frac{11}{81}$$
$$= \frac{25}{81}$$

4) The probability Mass function of a R.V X is defined as $P(X=0) = 3c^2$, $P(X=1) = 4c - 10c^2$, $P(X=2) = 5c - 1$ where $c > 0$ and $P(X=r) = 0$ if $r = 0, 1, 2$, find

i) the value of c

(iii) The distribution function of X.
(iv) The Largest value of X for which $F[x) < 1/2$

Soln

x	0	1	2
$P(x)$	$3c^2$	$4c-10c^2$	$5c-1$

i) To find c

$$\sum p(x) = 1$$

$$3c^2 + 4c - 10c^2 + 5c - 1 = 1$$

$$-7c^2 + 9c - 2 = 0$$

$$7c^2 - 9c + 2 = 0$$

$$(c-1)(7c-2) = 0$$

$$c = 1, 2/7$$

$$c \neq 1.$$

$$\therefore \boxed{c = 2/7}$$

(iv)

x	0	1	2
$P(x)$	$\frac{12}{49}$	$\frac{16}{49}$	$\frac{3}{7}$
$F[x)$	$\frac{12}{49}$	$\frac{28}{49}$	1

(ii) $P(0 < X < 2 / X > 0) = \frac{P[X=1] \cap P[X=1,2]}{P[X=1,2]}$

$$= \frac{P[X=1]}{P[X=1,2]} = \frac{\frac{16}{49}}{\frac{16}{49}+\frac{3}{7}}$$

$$= \frac{16}{49} \times \frac{49}{37}$$

$$= 16/37$$

(iii) $F[x] < 1/2$ is 0.

Continuous Random Variable:

X takes all its possible values in an interval.

Probability density function:

Let X be a continuous R.V then a function $f(x)$ is called pdf if

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cummulative distribution function

$$F[x] = P[X \leq x] = \int^{x} f(x)\,dx.$$

Note:

i) $f(x) = \frac{d}{dx} F[x]$

(ii) If X is continuous, then $P(a<x<b) = F[b] - F[a]$

1) Find the value of c given that pdf of a r.v X as $f(x) = \frac{c}{x^3}$, $1 < x < \infty$

Soln

W.K.T $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_{1}^{\infty} \frac{c}{x^3}\, dx = 1$$

$$c \int_{1}^{\infty} x^{-3}\, dx = 1$$

$$c \left[\frac{x^{-2}}{-2} \right]_{1}^{\infty} = 1$$

$$\frac{c}{-2} [0 - 1] = 1$$

$$\boxed{c = 2}$$

(2) If X is a discrete r.v taking the values 1, 2, 3, ... with probability function $P[X=x] = \frac{c^x}{x!}$, $x = 1, 2, \ldots$ then find the value of c.

Soln

$$\sum p_i = 1$$

$$\left[\frac{c^1}{1!}+\frac{c^2}{2!}+\frac{c^3}{3!}+\cdots\right]=1$$

$$\left[e^c-1\right]=1 \qquad \left[\because e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right]$$

$$e^c=2$$

$$\boxed{c=\log 2}$$

(8) If the pdf of a continuous r.v X is given by $f(x)=\begin{cases} ax, & 0\le x\le 1 \\ a, & 1\le x\le 2 \\ 3a-ax, & 2\le x\le 3 \\ 0, & \text{elsewhere}\end{cases}$

i) find the value of a

ii) find the cdf of X

iii) If x_1, x_2, x_3 are 3 independent observations of X, What is the probability that exactly one of these 3 is greater than 1.5 ?

Soln

i) To find a.

WKT $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_0^1 ax\,dx+\int_1^2 a\,dx+\int_2^3 (3a-ax)\,dx=1$$

$$a\left[\frac{x^2}{2}\right]_0^1+a\left[x\right]_1^2+\left[3ax-\frac{ax^2}{2}\right]_2^3=1$$

$$a\left[\tfrac{1}{2}-0\right]+a\left[2-1\right]+\left[9a-\frac{9a}{2}-6a+2a\right]=1$$

$$\frac{a}{2} + a + 5a - \frac{9a}{2} = 1$$

$$\frac{4a}{2} = 1$$

$$\boxed{a = 1/2}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(3-x), & 2 \le x \le 3 \\ 0, & \text{Otherwise} \end{cases}$$

ii) Cummulative distribution function of X is

$$F[x] = P[X \le x]$$

$$= \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

case (i) $x \le 0$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx = 0$$

case (ii) $0 \le x \le 1$

$$F[x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} \left(\frac{x}{2}\right) dx$$

$$= \left[0 + \frac{x^2}{4}\right]^{x} = \frac{x^2}{4}$$

case (iii) $\quad 1 \leq x \leq 2$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{x} \frac{1}{2}\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{1}{2}x\right]_1^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{x}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4}$$

Case (iv) $\quad 2 \leq x \leq 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{x} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x}{2}\right]_1^2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^x$$

$$= \left[\frac{1}{4} - 0\right] + \left[\frac{2}{2} - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{4} + \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - \frac{8}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

Case (v) $x \geq 3$

$$F[x] = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$F[x] = 1$$

$$F[x] = \begin{cases} 0 , & x \leq 0 \\ \frac{x^2}{4} , & 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{4} , & 1 \leq x \leq 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} , & 2 \leq x \leq 3 \\ 1 , & x > 3. \end{cases}$$

(iii) $P(1 \leq x \leq 2.5) = \int_{1}^{2.5} f(x)\,dx$

$$= \int_{1}^{2} f(x)\,dx + \int_{2}^{2.5} f(x)\,dx$$

$$= \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{2.5} \frac{1}{2}(3-x)\,dx$$

$$= \left[\frac{x}{2}\right]_1^2 + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_2^{2.5}$$

$$= \left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[\left(3(2.5) - \frac{5}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\left[\left(7.5 - \frac{5}{2}\right] - \left(\frac{8}{2}\right)\right]$$

$$= \frac{11}{16}.$$

iv $P(x > 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \frac{1}{2}(3-x)\,dx + 0$

$= \frac{1}{2}\left[x\right]_{1.5}^{2} + \frac{1}{2}\left[3x - \frac{x^2}{2}\right]_{2}^{3}$

$= \frac{1}{2}\left[2 - \frac{3}{2}\right] + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)\right]$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - 2\right]$

$= \frac{1}{4} + \left[\frac{9}{4} - 2\right] = \frac{1}{2}$

Assume $p = \frac{1}{2}$ $q = \frac{1}{2}$ $n = 3$

P[exactly one value greater than 1.5]

$= 3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)$

$= 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$

$= \frac{3}{8}$

4. A continuous random variable X has the pdf $f(x) = \begin{cases} \frac{K}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{Otherwise} \end{cases}$

i) find k (ii) Distribution function of X
iii) $P[X>0]$

Soln

i)
$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$K \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\pi/2 + \pi/2\right] = 1$$

$$k \cdot \pi = 1$$

$$k = 1/\pi$$

$$\therefore f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{Otherwise} \end{cases}$$

ii) In $-\infty < x < \infty$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}x + \tan^{-1}\infty\right]$$

$$F[x] = \frac{1}{\pi}\left[\tan^{-1}x + \pi/2\right]$$

iii $P[X>0] = 1 - P[X \le 0]$

$$= 1 - F[0]$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

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5) The Probability density function of a R.V X is given by
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x < 2 \\ 0, & \text{Otherwise} \end{cases}$$

i) Find the value of k

ii) Find $P(0.2 < x < 1.2)$

iii) What is $P(0.5 < X < 1.5 / X \ge 1)$

iv) Find the distribution function of X.

Soln

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

i) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 x\,dx + \int_1^2 k(2-x)\,dx = 1$$

$$\left[\frac{x^2}{2}\right]_0^1 + k\left[2x - \frac{x^2}{2}\right]_1^2 = 1$$

$$\left(\frac{1}{2}\right) + k\left[(4-2) - (2-\tfrac{1}{2})\right] = 1$$

$$\frac{1}{2} + k\left[2 - \tfrac{3}{2}\right] = 1$$

$$\tfrac{1}{2} + \tfrac{k}{2} = 1$$

$$\tfrac{k}{2} = 1 - \tfrac{1}{2}$$

$$\boxed{k = 1}$$

$$\therefore f(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 \leq x \leq 2 \\ 0 & , \text{Otherwise} \end{cases}$$

(iv) Distribution function

case (i) $x \leq 0$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = 0$$

Case (ii) $0 < x < 1$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx$$

$$= 0 + \int_{0}^{x} x\, dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

Case (iii) $1 < x < 2$

$$F[x] = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{x} f(x)\, dx$$

$$= 0 + \int_{0}^{1} x\, dx + \int_{1}^{x} (2-x)\, dx$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \tfrac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

Case (iv) $x \geq 2$

$$F[x] = \int_{-\infty}^{x} f(x)\, dx = 1$$

(ii) $P(0.2 < X < 1.2) = F(1.2) - F(0.2)]$

$$= \left[2(1.2) - \frac{(1.2)^2}{2} - 1\right] - \left[\cancel{2(0.2)} \frac{(0.2)^2}{2}\right)$$

$$= 0.68 - 0.02$$

$$= 0.66$$

(iii) $P(0.5 < X < 1.5 / X \geq 1) = \dfrac{P(0.5 < X < 1.5) \cap P(X \geq 1)}{P(X \geq 1)}$

$$= \frac{P(0.5 < X < 1.5) \cap (1 \leq x \leq 2)}{P(X \geq 1)}$$

$$= \frac{P(1 \leq X \leq 1.5)}{P(X \geq 1)}$$

$$= \frac{F(1.5) - F(1)}{1 - P(X < 1)} =$$

$0.875 - \tfrac{1}{2} = 0.375 \quad 0.75$

Mathematical Expectation

Let X be a r.v then the Mathematical expectation of X is given by

$$E[X] = \begin{cases} \sum x\, p(x), & X \text{ is discrete} \\ \int x\, f(x)\, dx, & X \text{ is continuous.} \end{cases}$$

Moments about origin

The r^{th} Moment about origin is

$$\mu_r' = E[X^r] = \begin{cases} \sum x^r p(x), & X \text{ is discrete} \\ \int x^r f(x)\, dx, & X \text{ is continuous} \end{cases}$$

$$\text{Mean} = E[X]$$

$$\text{Variance} = E[X^2] - \{E[X]\}^2$$

Moments about Mean [central Moments]

$$\mu_r = E[(X-\bar{X})^r] = \begin{cases} \sum (x-\bar{x})^r p(x), & X \text{ is discrete} \\ \int (x-\bar{x})^r f(x)\, dx, & X \text{ is continuous} \end{cases}$$

Discrete R.V	Continuous R.V
① $E[X] = \sum x\, p(x)$	$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$
② $E[X^r] = \mu_r' = \sum x^r p(x)$	$E[X^r] = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx$
③ Mean $= \mu_1' = \sum x\, p(x)$	Mean $\mu_1' = \int_{-\infty}^{\infty} x\, f(x)\, dx$
④ $\mu_2' = \sum x^2 p(x)$	$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$
⑤ Variance $= \mu_2' - {\mu_1'}^2$ $= E[X^2] - \{E[X]\}^2$	Variance $= \mu_2' - {\mu_1'}^2$ $= E[X^2] - \{E[X]\}^2$

Note:

1. $E[aX + b] = a\,E[X] + b$
2. $Var(aX + b) = a^2\, var\, X$
3. $cov(X, Y) = E[XY] - E[X]\,.\,E[Y]$
4. If X & Y are independent, then $cov(X, Y) = 0$
5. $cov(aX, bY) = ab\, cov(X, Y)$
6. $cov(X + a, Y + b) = cov(X, Y)$
7. $cov(aX + b, cY + d) = ac\, cov(X, Y)$
8. $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\, cov(X_1, X_2)$
9. $Var(X_1 - X_2) = Var(X_1) + Var\, X_2 - 2\, cov(X_1, X_2)$

Note:

1) $E[X+Y] = E[X] + E[Y]$

2) $E[XY] = E[X] \cdot E[Y]$.

(1) Given the following Probability distribution of X Compute (i) $E[X]$ (ii) $E[X^2]$ (iii) $E[2X \pm 3]$ (iv) $Var(2X \pm 3]$

X	-3	-2	-1	0	1	2	3
P(x)	0.05	0.10	0.30	0	0.30	0.15	0.10

Soln

(ii) $E[X] = \sum_{i=1}^{7} x_i P(x_i)$

$$= (-3)(0.05) + (-2)(0.1) + (-1)(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$$

$$= 0.25$$

(ii) $E[X^2] = \sum_{i=1}^{7} x_i^2 P(x_i)$

$$= (-3)^2(0.05) + (-2)^2(0.10) + (-1)^2(0.30) + 0 + 1^2(0.30) + 2^2(0.15) + 3^2(0.10)$$

$$= 2.95$$

(iii) $E[2X \pm 3] = 2E[X] \pm 3$

$$= 2(0.25) \pm 3$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{0} x^2 f(x)\, dx + \int_{0}^{1} x^2 \cdot x\, dx + \int_{1}^{2} x^2 (2-x)\, dx + \int_{2}^{\infty} x^2 f(x) dx$$

$$= 0 + \int_{0}^{1} x^3\, dx + \int_{1}^{2} (2x^2 - x^3)\, dx + 0$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{7}{6}$$

$$E[X^2] = 7/6$$

$$\text{Var}[X] = E[X^2] - [E[X]]^2$$

$$= \frac{7}{6} - 1 = \frac{1}{6}$$

<u>Moment</u> <u>generating</u> <u>function</u>:

$$M_X(t) = E[e^{tx}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & X \text{ is continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), & X \text{ is discrete.} \end{cases}$$

(1) Prove that r^{th} moment of the R.V 'X' about origin is $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$.

Soln

$$M_X(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^r}{r!} + \cdots\right]$$

$$= 1 + t\frac{E[X]}{1!} + t^2\frac{E[X^2]}{2!} + \cdots + \frac{t^r}{r!}E[X^r] + \cdots$$

$$= 1 + t\mu_1' + t^2\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

Note:

r^{th} Moment = co-efficient of $\frac{t^r}{r!}$

(2) Find μ_1' and μ_2' from $M_X(t)$.

Soln

WKT $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$$M_X(t) = \mu_0' + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

diff w.r.t 't'

$$M_X'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \cdots$$

$$M_x'(0) = \mu_1' = \text{Mean}$$

$$\therefore \text{Mean} = \mu_1' = M_x'(0) = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

lll^ly

$$M_x''(t) = \mu_2' + t\mu_3' + \dots$$

$$\mu_2' = M_x''(0) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}.$$

In General

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}.$$

(3) Obtain the Mgf of X about the pt $X = a$.

$$M_x(t) = E\left[e^{t(x-a)}\right]$$

$$= E\left[1 + \frac{t}{1!}(x-a) + \frac{t^2}{2!}(x-a)^2 + \dots + \frac{t^r}{r!}(x-a)^r + \dots\right]$$

$$= 1 + tE[x-a] + \frac{t^2}{2!}E(x-a)^2 + \dots + \frac{t^r}{r!}E(x-a)^r + \dots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots$$

$$[M(t)] = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{}$$

(4) Find the MGF of the random variable with the probability law $P(X=x) = q^{x-1} \cdot p \quad x = 1, 2, 3, \dots$ find the Mean and Variance.

Soln:

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} \, p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \, q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot \frac{p}{q}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q} (qe^{t}) \sum_{x=1}^{\infty} (qe^{t})^{x-1}$$

$$= pe^{t} \left[1 + qe^{t} + (qe^{t})^{2} + \dots \right]$$

$$= pe^{t} \left[1 - qe^{t} \right]^{-1}$$

M.g.F is $M_X(t) = \dfrac{pe^{t}}{1-qe^{t}}$

diff w.r.t 't'

$$\frac{d}{dt} M_X(t) = \frac{(1-qe^{t})pe^{t} - pe^{t}(-qe^{t})}{\quad}$$

$$= \frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2} \quad \text{—— (1)}$$

To find Mean:

$$\mu_1' \text{ (about origin)} = M_x'(0)$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\mu_1' = \text{Mean} = \frac{1}{p}$$

diff (1) w.r.t 't'

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t \, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)pe^t + 2pqe^{2t}\right]}{(1-qe^t)^4}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(t) = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$$

μ_2' (about origin) $= M_x''(0)$

$$\mu_2' = \frac{p(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\boxed{\text{Mean } \mu_1' = \frac{1}{p}}$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

5) find the MGF of the random Variable X having the probability density function $f(x) = \begin{cases} \frac{x}{4}e^{-x/2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Also deduce the first 4 Moments about the origin.

Soln

$$f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

MGF

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$u = x \quad v = e^{-(\frac{1}{2}-t)x}$

$$= \int_0^{\infty} e^{tx} \cdot \frac{x}{4} e^{-x/2}\, dx$$

$$= \frac{1}{4} \int_0^{\infty} x\, e^{-(\frac{1}{2}-t)} dt$$

$$= \frac{1}{4} \left[\frac{x\, e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)} - \frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[\frac{-2x\, e^{-(\frac{1-2t}{2})}}{(1-2t)} + \frac{4e^{(\frac{1-2t}{t})x}}{(1-2t)^2} \right]_0^{\infty}$$

$$= \frac{1}{4} \left[(0+0) - \left(0 + \frac{4e^0}{(1-2t)^2}\right) \right]$$

$$M_X(t) = \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

$$M_X'(t) = (-2)\,(1-2t)^{-3}\,(-2)$$
$$= 4(1-2t)^{-3}$$

$$M_X'(0) = 4$$

$$M_X''(t) = 4\,(-3)\,(1-2t)^{-4}\,(-2)$$
$$= 24(1-2t)^{-4}$$

$$M_X''(0) = 24$$

$$M_X'''(t) = 24\,(-4)\,(1-2t)^{-5}\,(-2)$$
$$= 192(1-2t)^{-5}$$

$$M_X'''(0) = 192$$

$$M_X^{IV}(t) = 192(-5)\,(1-2t)^{-6}\,(-2)$$
$$= 1920(1-2t)^{-6}$$

$$M_X^{IV}(0) = 1920.$$

6) Let X be a random variable with pdf

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

find
a) $P(X > 3)$
b) MGF of 'X'
c) $E[X]$ & $\operatorname{var} X$.

soln

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x > 0 \end{cases}$$

$M_x'(t) = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$

$E[x] = \text{Mean} = M_x'(0) = 3$

$M_x''(t) = -6(1-3t)^{-3} = 18(1-3t)^{-3}$

$M_x''(0) = 18$

$E[x^2] = 18$

$\text{Var } x = E[x^2] - [E[x]]^2$

$= 18 - 3^2$

$\text{Var } x = 9.$

7) A continous random variable X has the pdf $f(x) = kx^2e^{-x}$, $x \geq 0$ find the r^{th} moment of X about the origin. Hence find the variance of X.

Soln

To find k

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$$k\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_0^{\infty}=1$$

$u=x^2 \quad v=e^{-x}$

$u'=2x \quad v_1=-e^{-x}$

$u''=2 \quad v_2=e^{-x}$

$u'''=0 \quad v_3=-e^{-x}$

$$k\left[(\infty-\infty-2e^{-\infty})-(0-0-2e^{-0})\right]=1$$

$$2k=1$$

$$k=\frac{1}{2}$$

$$f(x)=\frac{1}{2}x^2e^{-x}.$$

find r^{th} moment:

$$\mu_r'=E[x^r]=\int_{-\infty}^{\infty}x^r f(x)\,dx$$

$$=\int_0^{\infty}x^r\frac{1}{2}x^2e^{-x}\,dx$$

$$=\frac{1}{2}\int_0^{\infty}x^{r+2}e^{-x}\,dx$$

$u=x^{r+2} \quad v=e^{-x}$

$u'=(r+2)x^{r+1} \quad v_1=-e^{-x}$

$u''=(r+2)(r+1)x^{r} \quad v_2=e^{-x}$

$v_3=-e^{-x}$

$$=\frac{1}{2}\left[-e^{-x}x^{r+2}-(r+2)x^{r+1}e^{-x}-(r+2)(r+1)x^{r}e^{-x}-\cdots\cdots-(r+2)!e^{-x}\right]_0^{\infty}$$

$$=-\frac{1}{2}\left[e^{-x}\left(x^{r+2}+(r+2)x^{r+1}+\cdots\cdots(r+2)!\right)\right]_0^{\infty}$$

$$\mu_r'=-\frac{1}{2}\left(-(r+2)!\right)=\frac{1}{2}(r+2)!$$

$$\mu_1'=\frac{3!}{2}=3 \qquad Var(x)=E[x^2]-[E[x]]^2$$

$$=12-3^2$$

$$\mu_2'=\frac{4!}{2}=12 \qquad =3$$

8) If the pdf of 'X' is given by

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Show that $E[X^r] = \frac{2}{(r+1)(r+2)}$

b) Using this result, evaluate $E[(2X+1)^2]$

Soln

$$f(x) = 2(1-x)$$

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= \int_0^1 x^r 2(1-x)\, dx$$

$$= 2\int_0^1 (x^r - x^{r+1})\, dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - (0-0)\right]$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right]$$

$$= 2\left[\frac{\cancel{r}+2-\cancel{r}-1}{(r+1)(r+2}\right]$$

$$E[X^r] = \frac{2}{(r+1)(r+2)}$$

Put $r = 1$

$$E[X] = \frac{2}{(1+1)(1+2)} = \frac{1}{3}$$

$$E[X^2] = \frac{2}{(2+1)(2+2)} = \frac{1}{6}$$

$$E[(2x+1)^2] = E[4x^2+1+4x]$$

$$= E[4x^2] + E[1] + E[4x]$$

$$= 4E[x^2] + 4E[x] + 1$$

$$= \frac{4}{6} + \frac{4}{3} + 1$$

$$= \frac{4+8+6}{6}$$

$$= \frac{18}{6}$$

$E[(2x+1)^2]$

9 consider a discrete r.v 'x' with probability function $p(X=x) = \begin{cases} \frac{1}{x(x+1)}, & x = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$

Show that $E[x]$ does not exist even though MGF exist.

Soln

$$p(x) = \frac{1}{x(x+1)}$$

Gn $E[X] = \sum_{x=1}^{\infty} x\, p(x)$

$$= \sum_{x=1}^{\infty} \cancel{x} \frac{1}{x(x+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1 + \sum_{x=1} \frac{1}{x}$$

$\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series.

$\therefore E[X]$ does not exist and hence no moment exists.

Now, MGF of X is gn by

$$M_X(t) = \sum_{x=1} e^{tx} p(x) = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

Put $z = e^t$

$$= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)}$$

$$= \frac{z}{1\cdot 2} + \frac{z^2}{2\cdot 3} + \frac{z^3}{3\cdot 4} + \cdots$$

$$= z\left(1 - \frac{1}{2}\right) + z^2\left(\frac{1}{2} - \frac{1}{3}\right) + z^3\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= \left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right] - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} - \cdots$$

$$= -\log(1-z) - \frac{1}{z}\left[\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots\right)$$

$$= -\log(1-z) - \frac{1}{z}\left[-z + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right]$$

$$= -\log(1-z) + 1 - \frac{1}{z}\left[-\log(1-z)\right]$$

$= 1 + \left(\frac{1}{z} - 1\right) \log(1-z)$

$M_X(t) = 1 + (e^{-t} - 1)\log(1 - e^{t}), \; t < 0$

$M_X(t) = 1, \; \text{for } t = 0.$

$M_X(t)$ does not exist for $t > 0$

10) A random variable X has pdf

$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ find the MGF

when $t < 2$. find the first 4 Moments about the origin.

Soln

MGF of X is

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} 2e^{-2t} e^{tx}\, dx = 2\int_{0}^{\infty} e^{-(2-t)x}\, dx$$

$$= 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{-2}{2-t}\left[e^{-\infty} - e^{0}\right]$$

$$M_X(t) = \frac{2}{2-t}$$

Gn $t < 2$

$\frac{t}{2} < 1$

$\left|\frac{t}{2}\right| < 1$

$$M_X(t) = \frac{2}{2(1-t)} = \left(1 - \frac{t}{2}\right)^{-1}$$

$$= 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \cdots \quad \because \left|\frac{t}{2}\right| < 1$$

$$= 1 + \frac{1}{2}\frac{t}{1!} + \frac{2!}{4}\cdot\frac{t^2}{2!} + \frac{3!}{8}\frac{t^3}{3!} + \frac{4!}{16}\frac{t^4}{4!} + \cdots$$

$$\mu_1' = \frac{1}{2}$$

$$\mu_2' = \frac{2!}{4} = \frac{1}{2}$$

$$\mu_3' = \frac{3!}{8} = \frac{3}{4}$$

$$\mu_4' = \frac{4!}{2^4} = \frac{24}{16} = \frac{3}{2}$$

Note:

If the MGF of X is

$$M_x(t) = 1 + m_1\frac{1}{1!} + m_2\frac{t^2}{2!} + m_3\frac{t^3}{3!} + \cdots$$

μ_1' = Co-efficient of $\frac{t}{1!}$

& μ_2' = Co-efficiet of $\frac{t^2}{2!}$

⋮

$= 0.608$

(iv) $P[\text{B/w } 1 \text{ and } 3 \text{ defectives}] = P[1 \le X \le 3]$

$= P[X=1] + P[X=2] + P[X=3]$

$= \Big[\cancel{\left(20C_0 (1/10)^0 (9/10)^{20}\right)} + \left(20 C_1 (1/10)^1 (9/10)^{19}\right)$

$+ \left(20 C_2 (1/10)^2 (9/10)^{18}\right) + \left(20 C_3 (1/10)^3 (9/10)^{17}\right) \Big]$

$= [0.27 + 0.28517 + 0.1901178]$

$= 0.7452$

18.1.13

Poisson distribution:

A random variable 'x' is said to follow poisson distribution, if its probability mass function is

$$P[X=x] = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2, 3 \ldots$$

Moment Generating Function:

$M_x(t) = E[e^{tx}]$

$= \sum_{x=0}^{\infty} e^{tx} p(x)$

$= \sum^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^{x}}{}$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[\frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \left[1 + \frac{(\lambda e^t)}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \right]$$

$$= e^{-\lambda} \left[e^{\lambda e^t} \right]$$

$$= e^{\lambda [e^t - 1]}$$

$$\boxed{\therefore M_x(t) = e^{\lambda(e^t - 1)}}$$

Mean:

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$M_x'(t) = e^{\lambda(e^t - 1)} \cdot \lambda e^t$$

$$M_x'(0) = e^{\lambda(e^0 - 1)} \cdot \lambda e^0$$

$$= e^0 \cdot \lambda e^0$$

$$= \lambda$$

$$\boxed{\therefore \text{Mean} = \lambda} = E[x]$$

To find $E[x^2]$:

$$M_x''(t) = \lambda\left[e^{\lambda(e^t-1)} \cdot e^t + e^t \cdot e^{\lambda(e^t-1)} \cdot \lambda e^t\right]$$

$$M_x''(0) = \lambda\left[e^{\lambda(e^0-1)} \cdot e^0 + e^0 \cdot e^{\lambda(e^0-1)} \cdot \lambda e^0\right]$$

$$= \lambda\left[e^0 \cdot e^0 + e^0 \cdot e^0 \cdot \lambda e^0\right]$$

$$= \lambda[1+\lambda]$$

$$= \lambda + \lambda^2$$

$$E[x^2] = \lambda + \lambda^2$$

Variance:

$$Var[x] = E[x^2] - [E[x]]^2$$

$$= \lambda + \cancel{\lambda^2} - \cancel{\lambda^2}$$

$$= \lambda$$

$$\boxed{\therefore Var[x] = \lambda}$$

Note:

In poisson distribution,

Mean = Variance = λ

1\. The atoms of a radioactive elements are disintegrating. If every gram of this elements an average emits 3.9 alphaparticles per second. What is the probability that during the next second the number of alpha particles emitted from 1 gram is

(i) atmost 6

(ii) atleast 2

(iii) atleast 3 and atmost 6

Solution:

Given $\lambda = 3.9$

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

(i) $P[\text{atmost } 6] = P[X \leq 6]$

$$= P[X=0]+P[X=1]+P[X=2]+P[X=3]+ P[X=4]+P[X=5]+P[X=6]$$

$$= e^{-3.9}\left[\frac{(3.9)^0}{0!}+\frac{(3.9)^1}{1!}+\frac{(3.9)^2}{2!}+\frac{(3.9)^3}{3!}+ \frac{(3.9)^4}{4!}+\frac{(3.9)^5}{5!}+\frac{(3.9)^6}{6!}\right]$$

$= 0.0202 [$

$= 0.0202 \left[44.4365 \right]$

$= 0.8976$

(ii) p(atleast 2) $= P[x \geq 2]$

$= 1 - P[x < 2]$

$= 1 - P[x=0] + P[x=1]$

$= 1 - e^{-3.9}\left[1 + 3.9\right]$

$= 1 - 0.0202\left[4.9\right]$

$= 1 - 0.09898$

$= 0.90102$

(iii) P[atleast 3 and atmost 6] $= P[3 \leq x \leq 6]$

$= P[x=3] + P[x=4] + P[x=5] + P[x=6]$

$= 0.0202 \left[9.8865 + 9.6393 + 7.5186 + 4.8871\right]$

$= 0.0202 \left[31.9315\right]$

$= 0.6450$

2\. Suppose that the number of calls coming into telephone exchange b/w 9.am and 10 am is a poisson random variable with parameter 2, and the number of telephone calls coming b/w 10 AM and 11 AM is a random variable with parameter 6. If these two random variables are independent. What is the probability that more than 5 calls come in between 9 AM and 11 AM.

Solution:

Let X_1 - calls b/w 9.AM and 10 AM with $\lambda_1 = 2$

X_2 - calls b/w 10 AM and 11 A.M with $\lambda_2 = 6$

WKT,

$$X = X_1 + X_2.$$

$$\lambda = \lambda_1 + \lambda_2$$

$$\therefore \lambda = 2 + 6 = 8$$

$$\boxed{\lambda = 8}$$

X - calls b/w 9 AM and 11 AM with $\lambda = 8$

$$P[X = x] = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$P[X > 5] = 1 - P[X \leq 5]$$

$$= 1 - \Big[P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) \Big]$$

$$= 1 - e^{-8}\left[\frac{8^0}{0!} + \frac{8^1}{1!} + \frac{8^2}{2!} + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!}\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[1 + 8 + 32 + 85.33 + 170.666 + 273.06\right]$$

$$= 1 - 3.354 \times 10^{-4}\left[570.056\right]$$

$$= 1 - 0.1911$$

$$= 0.808$$

29. 3. The MGF of a random variable X be $e^{4(e^t-1)}$. Show that $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.93$

Solution:

Given,

$$MGF = e^{4(e^t-1)} = M_X(t) \quad \text{---} (1)$$

In poisson distribution.

$$MGF = M_X(t) = e^{\lambda(e^t-1)} \quad \text{---} (2)$$

On comparing (1) & (2)

$$\lambda = 4$$

In poisson distribution

Mean = Variance = λ

$$\mu = \sigma^2 = 4$$

To prove :

$$P[\mu - 2\sigma < X < \mu + 2\sigma] = 0.93$$

LHS

$$P[4 - 2(2) < X < 4 + 2(2)] = P[0 < x < 8)$$

$$= P[x=1] + P[x=2] + P[x=3] + P[x=4] +$$

$$P[x=5] + P[x=6] + P[x=7)$$

$$= e^{-4}\left[\frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right)$$

$$= 0.01831\left[4 + 8 + 10.66 + 10.666 + 8.533 + 5.6888 + 3.2507\right]$$

$$= 0.01831\left[50.8045\right)$$

$$= 0.93$$

---

30. A. If X is a poisson random Variable such that $P[x=2] = 9P[x=4] + 90P[x=6]$

Find the Variate.

Solution:

$$P[x=x] = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P[x=2] = 9\,P[x=4] + 90\,P[x=6]$$

$$\frac{e^{-\lambda}\lambda^2}{} \quad \frac{e^{-\lambda}\lambda^{4}}{} + 90\,\frac{e^{-\lambda}\lambda^{6}}{}$$

$$\frac{1}{1\times 2!} = \frac{3\lambda^2}{1\times 2\times 3\times 4} + \frac{30\lambda^4}{1\times 2\times 3\times 4\times 5\times 6}$$

$$1 = \frac{3\lambda^2}{4} + \frac{\lambda^4}{4}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$(\lambda^2)^2 + 3(\lambda^2) - 4 = 0$$

$$(\lambda^2 - 1)(\lambda^2 + 4) = 0$$

$\lambda^2 = 1 \qquad \lambda^2 + 4 = 0$

$\lambda = 1 \qquad \lambda^2 = -4$

$\lambda^2 = -4$ not possible

$\therefore \lambda = 1$

Variance $= \lambda = 1$

28.1.13

Geometric distribution

A random variable 'X' is said to follow geometric distribution if its probability mass function is

$$P[X=x] = q^{x-1}p, \quad x = 1, 2$$

MGF :

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot q^{-1} \cdot p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= \frac{p}{q} \left[(qe^{t}) + (qe^{t})^{2} + (qe^{t})^{3} + \cdots \right]$$

$$= \frac{p}{q} \cdot qe^{t} \left[1 + (qe^{t}) + (qe^{t})^{2} + \cdots \right]$$

$$= pe^{t} \left[1 - qe^{t} \right]^{-1}$$

$$\boxed{M_X(t) = \frac{pe^{t}}{1 - qe^{t}}}$$

Mean:

$$M_x(t) = \frac{Pe^t}{1-qe^t}$$

$$M_x'(t) = \frac{(1-qe^t)Pe^t - Pe^t(-qe^t)}{(1-qe^t)^2}$$

$$= \frac{Pe^t - \cancel{pq}e^{2t} + p\cancel{q}e^{2t}}{(1-qe^t)^2}$$

~~$M_x'(0) = Pe^0 - pqe^0$~~

$$M_x'(t) = \frac{Pe^t}{(1-qe^t)^2}$$

$$M_x'(0) = \frac{Pe^0}{(1-qe^0)^2}$$

$$= \frac{P}{(1-q)^2}$$

$$= \frac{P}{P^2}$$

$$= \frac{1}{P}$$

$\therefore$ Mean $= E[x] = \frac{1}{P}$

To find : $E[x^2]$

$$M_x'(t) = \frac{Pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 Pe^t - Pe^t(2)(1-qe^t)(-qe^t)}{}$$

$$= \frac{(1-qe^t)\left[(1-qe^t)\,pe^t + 2pqe^{2t}\right]}{(1-qe^t)^{\cancel{4}\,3}}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}$$

$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{pe^0 + pqe^0}{(1-qe^0)^3}$$

$$= \frac{p+pq}{(1-q)^3}$$

$$= \frac{p(1+q)}{p^{\cancel{3}2}}$$

$$= \frac{1+q}{p^2}$$

$$M_x''(0) = E[x^2] = \frac{1+q}{p^2}$$

Variance:

$$\mathrm{Var}[x] = E[x^2] - (E[x])^2$$

$$= \frac{1+q}{p^2} - \left(\frac{1}{p}\right)^2$$

$$= \frac{\cancel{1}+q-\cancel{1}}{p^2}$$

$$= \frac{q}{p^2}$$

Memoryless property :-

The memoryless property is given by $p[x > s+t / x > s] = P[x > t]$ for any $s, t > 0$.

Consider :

$$P[x > s+t] = \sum_{x = s+t+1}^{\infty} p q^{x-1}$$

$$= p\left[q^{s+t} + q^{s+t+1} + q^{s+t+2} + q^{s+t+3} + \cdots\right]$$

$$= p q^{s+t}\left[1 + q + q^2 + \cdots\right]$$

$$= p q^{s+1} [1-q]^{-1}$$

$$= \frac{p q^{s+1}}{1-q}$$

$$P[x > s+t] = \frac{p q^{s+t}}{1-q}$$

$$= \frac{p q^{s+t}}{p}$$

$$= q^{s+t}$$

$$\therefore P[x > s+t] = q^{s+t} \quad \text{—— (1)}$$

$$\therefore P[x > s] = q^{s} \quad \text{—— (2)}$$

$$P[x > t] = q^{t} \quad \text{—— (3)}$$

$$P[x>s+t/x>s] = \frac{P[x>s+t \cap x>s]}{P[x>s]}$$

$$= \frac{P[x>s+t]}{P[x>s]}$$

$$= \frac{q^{s+t}}{q^{s}}$$

$$= q^{t}$$

$$= P[x>t]$$

$$\therefore P[x>s+t/x>s] = P[x>t]$$

1. Let one copy of a magazine out of 10 copies bears a special price following distribution determine its mean and variance.

Solution:

Given

$P = \frac{1}{10}$

$\therefore P+q=1$

$q=1-P$

$=1-\frac{1}{10}$

$\therefore q=\frac{9}{10}$

Mean:

$= 10$

$\therefore$ Mean $= 10$

Variance:

$$Var[x] = \frac{q}{p^2}$$

$$= \frac{9}{10} \Big/ \left(\frac{1}{10}\right)^2$$

$$= \frac{9 \times 100}{10}$$

$$= 90$$

$\therefore$ Variance $= 90$.

Uniform distribution [rectangular distribution]

A random variable x is said to have a continuous uniform distribution, if its probability density function is given by.

$$f(x) = \begin{cases} 1/b-a, & a < x < b \\ 0 & \text{otherwise}. \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{(b-a)t}\left[e^{bx} - e^{ax}\right]$$

$$= \frac{e^{bx} - e^{ax}}{(b-a)t}$$

$$M_X(t) = \frac{e^{bx} - e^{ax}}{(b-a)t}$$

Mean:

$$E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2)$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

$\therefore E[X] = \text{Mean} = \frac{b+a}{}$

Variance:

To find $E[x^2]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}.$$

$$E[x^2] = \frac{a^2+ab+b^2}{3}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a^2+b^2+2ab)}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{}$$

32. 1. If X is uniformly distributed over $(0,10)$ find the probability that

(i) $(X<2)$

(ii) $(X>8)$

(iii) $(3<X<9)$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x < b \\ 0, & \text{Otherwise} \end{cases}$$

$(0,10)$

$$f(x) = \begin{cases} \frac{1}{10}; & 0 < x < 10 \\ 0; & \text{Otherwise.} \end{cases}$$

(i) $P[X<2]$

$$f(x) = \int_0^2 f(x)\,dx$$

$$= \int_0^2 \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left[x\right]_0^2$$

$$= \frac{1}{10}[2-0]$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

$\therefore P[X<2] = 1/5$

(ii) $P[x > 8]$

$$f(x) = \int_{8}^{10} \frac{1}{10}\, dx$$

$$= \frac{1}{10}\left[x\right]_8^{10}$$

$$= \frac{1}{10}\left[10-8\right]$$

$$= \frac{2}{10}$$

$$= \frac{1}{5}$$

$\therefore P[x > 8] = \frac{1}{5}$

(iii) $P[3 < x < 9]$

$$f(x) = \int_{3}^{9} \frac{1}{10}\, dx$$

$$= \frac{1}{10}\left[x\right]_3^{9}$$

$$= \frac{(9-3)}{10}$$

$$= \frac{6}{10}$$

$$= 3/5$$

$\therefore P[x > 8] = 3/5.$

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2. If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha < 0$ find α, so ~~show~~ that

(i) $P[X > 1] = \frac{1}{3}$

(ii) $P[|X| < 1) = P(|X| > 1)$

Solution:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{Otherwise.} \end{cases}$$

$a = -\alpha$, $b = \alpha$.

$$f(x) = \begin{cases} \frac{1}{2\alpha}, & -\alpha < x < \alpha \\ 0, & \text{Otherwise.} \end{cases}$$

(i) $P[X > 1] = \frac{1}{3}$

$$\int_1^\alpha \frac{1}{2\alpha}\, dx = \frac{1}{3}$$

$$\frac{1}{2\alpha}\left[x\right]_1^\alpha = \frac{1}{3}$$

$$\frac{1}{2\alpha}\left[\alpha - 1\right] = \frac{1}{3}$$

$$3\alpha - 3 = 2\alpha$$

$$3\alpha - 2\alpha - 3 = 0$$

$$\alpha - 3 = 0$$

$$\alpha = 3$$

$$\therefore \boxed{\alpha = 3}$$

(ii) $P[|x|<1] = P[|x|>1]$

$$P[|x|<1] = 1-P(|x|<1)$$

$$2P[|x|<1] = 1$$

$$2P[-1<x<1] = 1$$

$$2\int_{-1}^{1}\frac{1}{2\alpha}\,dx = 1$$

$$\frac{2}{2\alpha}\left[x\right]_{-1}^{1} = 1$$

$$\frac{1}{\alpha}\left[1-(-1)\right] = 1$$

$$\frac{1}{\alpha}[2] = 1$$

$$\boxed{\alpha = 2}$$

---

34. 3. Four buses arrive at a specified stop at 15 min intervals starting at 7AM. (ie) they arrive at 7, 7.15, 7.30, 7.45. AM and so on. If a passenger arrives at a time (ie) uniformly distributed between 7 and 7.30 AM. Find the probability that he waits

(a) less than 5 mins for a bus.

(b) more than 10 mins for a bus.

Solution:

Let 'x' denote the number of minutes passed 7, that the passenger arrived busstop in (0, 30)

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{Otherwise.} \end{cases}$$

(i) A passenger will have to wait less than 5 minutes, if he arrives between 7.10 and 7.15 and if he arrives between 7.25 and 7.30.

$P(10 < x < 15) + P(25 < x < 30)$

$$= \int_{10}^{15} \frac{1}{30}\, dx + \int_{25}^{30} \frac{1}{30}\, dx$$

$$= \frac{1}{30}\left[[x]_{10}^{15} + [x]_{25}^{30} \right]$$

$$= \frac{1}{30}\left[(15-10) + (30-25) \right]$$

$$= \frac{1}{30}\left[5+5 \right]$$

$$= \frac{10}{30}$$

$$= \frac{1}{3}$$

$$\therefore \left[P[10 < x < 15) + P(25 < x < 30) \right] = \frac{1}{3}$$

(ii) A passenger will have to wait more than ten minutes if he arrives b/w 7 and 7.05, (or) b/w 7.15 and 7.20.

$P[0 < X < 5] + P[15 < X < 20]$

$= \int_0^5 \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx$

$= \frac{1}{30}\left[[x]_0^5 + [x]_{15}^{20}\right]$

$= \frac{1}{30}\left[(5-0) + (20-15)\right]$

$= \frac{1}{30}\left[5+5\right]$

$= \frac{10}{30}$

$= \frac{1}{3}$

$P[0 < X < 5] + P[15 < X < 20] = \frac{1}{3}$.

24/1/'13

Exponential distribution

A continuous random variable x is said to follow an exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

MGF:

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{tx} e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} \left[e^{-(\lambda - t)x}\, dx \right]$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda-t)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{\lambda}{\lambda - t}$$

$$M_x(t) = \frac{\lambda}{\lambda - t}, \ \lambda \geq t$$

Mean:

$$M_x(t) = \lambda (\lambda - t)^{-1}$$

$$M_x'(t) = \lambda (-1) (\lambda - t)^{-2} \cdot (-1)$$

$$= \lambda (\lambda - t)^{-2}$$

$$= \frac{\lambda}{(\lambda - t)^2}$$

$$M_x'(0) = \frac{\lambda}{(\lambda - 0)^2}$$

$$= \frac{\lambda}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

$$\boxed{\therefore \text{Mean} = E[x] = \frac{1}{\lambda}}$$

To find $E[x^2]$:

$$M_x'(t) = \lambda (\lambda - t)^{-2}$$

$$\cancel{E[x^2]} \ M_x''(t) = \lambda (-2) (\lambda - t)^{-3} \cdot (-1)$$

$$= \frac{2\lambda}{(\lambda - t)^3}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2-1}{\lambda^2}$$

$$= 1/\lambda^2$$

$$\therefore \boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Memoryless property:

If x is exponentially distributed, then $P[x > s+t / x > s]$ $= P(x > t)$ for any $s, t > 0$.

Solution:

$$P[x > k) = \int_k^\infty f(x)\, dx.$$

$$= \int_k^\infty \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_k^\infty e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$$

$$= \frac{\lambda}{-\lambda}\left[-e^{-\lambda k}\right]$$

$$= e^{-\lambda k}.$$

$$P[x > s+t / x > s) = \frac{P[(x > s+t) \cap (x > s)]}{P[x > s]}$$

$$= \frac{P[x > s+t]}{P[x > s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda s - \lambda t} \cdot e^{\lambda s}$$

$$= e^{-\lambda s - \lambda t + \lambda s}$$

$$= e^{-\lambda t}$$

$$= P[x > t]$$

$$\therefore P[x > s+t / x > s] = P[x > t]$$

35. i. The time (in hrs) required to repair a machine is exponentially distributed with parameter $\lambda = 1/2$

(a) what is the probability that the

Given that its duration exceeds 8 Hrs.?

Solution:

Let 'x' be the random Variable which represents the time to repair the machine.

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & x > 0. \end{cases}$$

(i) $P[X > 2] = e^{-\lambda 2}$ $\quad [\because P[X > k] = e^{-\lambda k}]$

$= e^{-2/2}$

$= e^{-1}$

$= 0.3678$

(ii) $P[X \geq 11 / X > 8] = P[X > 8+3 / X > 8]$

$= P[X > 3]$

$\because P[X > s+t / X > s] = P[X > t]$

$= e^{-3\lambda}$

$= e^{-3/2}$

$= 0.2231$

25/11/13 36 2. If x is a random variable which follows an exponential distribution with parameter λ with $P[x \le 1) = P[x > 1]$ Find Variance of x.?

Solution:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$$P(x \le 1) = P(x > 1)$$

$$1 - P(x > 1) = P(x > 1)$$

$$2P(x > 1) = 1$$

$$P(x > 1) = 1/2$$

$$e^{-\lambda} = 1/2 \quad \left[\because P[x > k] = e^{-\lambda k}\right]$$

$$\frac{1}{e^{\lambda}} = \frac{1}{2} \Rightarrow e^{\lambda} = 2$$

Taking log on both sides,

$$\lambda = \log_e 2$$

$$Var[x] = \frac{1}{\lambda^2}$$

$$= \frac{1}{(\log_e 2)^2}$$

$$\therefore Var[x] = \frac{1}{(\log_e 2)^2}$$

Gamma distribution:

The continuous random variable 'x' is said to follow a Gamma distribution with parameter λ, if its probability function is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma\lambda} & \lambda>0,\ 0<x<\infty \\ 0 & , \text{otherwise} \end{cases}$$

MGF:

$$M_X(t) = E[e^{tx}]$$

$$= \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma\lambda}\, dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{tx} e^{-x} x^{\lambda-1}\, dx$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\, dx$$

Put

$u = (1-t)x$ | $x \to 0 \Rightarrow u \to 0$

$du = (1-t)\,dx$ | $x \to \infty \Rightarrow u \to \infty$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty e^{-u} \left(\frac{u}{1-t}\right)^{\lambda-1} \cdot \frac{du}{1-t}$$

$$= \frac{1}{\Gamma\lambda} \int_0^\infty \frac{e^{-u}\, u^{\lambda-1} \cdot du}{(1-t)^{\lambda-1}(1-t)}$$

$$= \frac{1}{\Gamma\lambda} \cdot \frac{1}{(1-t)^\lambda} \int_0^\infty e^{-u}\, u^{\lambda-1}\, du.$$

$$= \frac{1}{\cancel{\Gamma\lambda}(1-t)^\lambda} \cdot \cancel{\Gamma\lambda} \quad \left[\because \Gamma n = \int_0^\infty e^{-x}\, x^{n-1} \cdot dx\right]$$

$$= \frac{1}{(1-t)^\lambda}$$

$$\therefore M_x(t) = \frac{1}{(1-t)^\lambda}$$

Mean:

$$M_x(t) = (1-t)^{-\lambda}$$

$$M_x'(t) = -\lambda(1-t)^{-\lambda-1} \cdot (-1)$$

$$= \lambda(1-t)^{-(\lambda+1)}$$

$$= \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$M_x'(0) = \frac{\lambda}{(1-0)^{\lambda+1}}$$

$$= \frac{\lambda}{(1)^{\lambda+1}} = \frac{\lambda}{1} = \lambda$$

$E[x^2]$:

$$M_x'(t) = \frac{\lambda}{(1-t)^{\lambda+1}}$$

$$= \lambda(1-t)^{-(\lambda+1)}$$

$$M_x''(t) = \lambda\,.\,(-(\lambda+1))\,(1-t)^{-(\lambda+2)}\,(-1)$$

$$= \lambda(\lambda+1)(1-t)^{-(\lambda+2)}$$

$$M_x''(0) = \lambda(\lambda+1)(1-0)^{-(\lambda+2)}$$

$$= \lambda(\lambda+1)$$

$$= \lambda^2+\lambda$$

$$E[x^2] = \lambda^2+\lambda$$

Variance:

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \cancel{\lambda^2}+\lambda - \cancel{\lambda^2}$$

$$= \lambda$$

$\therefore Var[x] = \lambda$

Normal distribution:

A continuous random variable x is said to follow a normal distribution with mean μ, and variance σ^2, if its density function is given by the probability law,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}, \quad -\infty < x < \infty,$$

$$\sigma > 0, \ -\infty < \mu < \infty.$$

MGF:

$M_x(t) = E[e^{tx}]$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Put $z = \frac{x-\mu}{\sigma}$ | $x \to -\infty \Rightarrow z \to -\infty$

$dz = \frac{dx}{\sigma}$ | $x \to \infty \Rightarrow z \to \infty$

$\sigma\, dz = dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \cdot e^{-\frac{z^2}{2}} \times \sigma\, dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2t\sigma z)}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left(z^2 - 2t\sigma z + (t\sigma)^2\right) + \frac{(t\sigma)^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2} e^{\frac{t^2\sigma^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2} dz$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{(\sqrt{2\pi})} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\cancel{\sqrt{2\pi}}} \cdot \cancel{\sqrt{2\pi}}$$

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

<u>Mean:</u>

$$M_x(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}.$$

$$M_x'(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$= \mu$

$$\boxed{\text{Mean} = E[X] = \mu.}$$

To find $E[X^2]$:

$$M_X''(t) = e^{\mu t + \frac{t^2\sigma^2}{2}} \cdot \sigma^2 + (\mu + t\sigma^2)\, e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + t\sigma^2)$$

$$M_X''(0) = e^{0} \cdot \sigma^2 + (\mu + 0)\, e^{0} (\mu + 0)$$

$$E[X^2] = \sigma^2 + \mu^2$$

Variance:

$$\text{Var}[X] = E[X^2] - (E[X])^2$$

$$= \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2}$$

$$= \sigma^2$$

$$\therefore \boxed{\text{Variance} = \sigma^2}$$

Standard normal distribution.

$$Z = \frac{X - \mu}{\sigma} \quad \text{with parameter}$$

μ and σ

Basic properties:

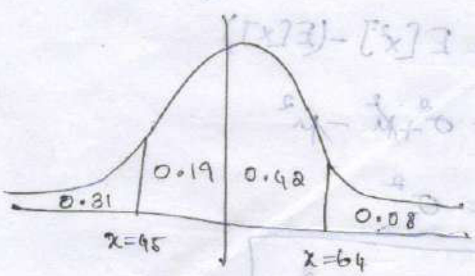
* Total area under the standard normal curve is equal to 1.

* The standard normal curve is asymptotic to x-axis.

The standard normal curve is symmetric about zero, most of the area under the standard normal curve lies b/w -3 and 3.

37 1. In a normal distribution 31 % of the items are ~~under~~ under ~~45~~ 45 and 8 % are ~~under~~ over 64. Find the mean and standard deviation.

Solution:



~~The value of z corresponding to the area 0.19 is~~

Let the mean and standard deviation of the given normal distribution be μ and σ.

The value of z corresponding to the area 0.19 is 0.5 nearly

$$\frac{45-\mu}{\sigma} = -0.5$$

$$-0.5\sigma + \mu = 45 \quad \text{———(1)}$$

The value of z corresponding to the area 0.42 is 1.4 nearly.

$$\frac{64-\mu}{\sigma} = 1.4$$

$$1.4\sigma + \mu = 64 \quad \text{——} (2)$$

Solve (1) & (2)

$$-0.5\sigma + \mu = 45$$
$$1.4\sigma + \mu = 64$$
$$(-) \quad (-) \quad (-)$$
$$-1.9\sigma = -19$$

$$\sigma = \frac{-19}{-1.9}$$

$$\sigma = 10$$

Sub $\sigma = 10$ in (2)

$$(1.4)10 + \mu = 64$$

$$14 + \mu = 64$$

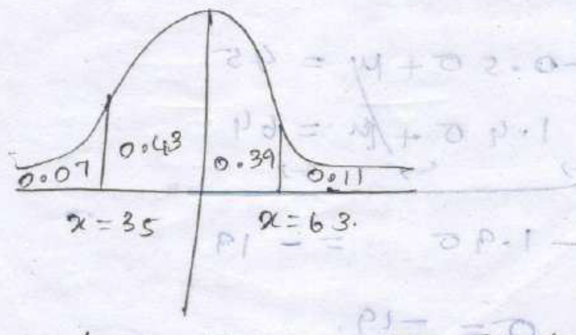
$$\mu = 64 - 14$$

$$\mu = 50$$

$$\therefore \sigma = 10, \mu = 50.$$

2. In a distribution, exactly normal 7% of the items are under 35 and 39% items are under 63. What are the mean and Standard deviation of the distribution.

Solution:



Let the mean and Standard deviation of the normal distribution be μ and σ.

The Value of z corresponding to the area 0.43 is 1.47 nearly.

$$\frac{35-\mu}{\sigma} = -1.47$$

$$-1.47\sigma + \mu = 35 \quad \text{——} (1)$$

The Value of z corresponding to the area 0.39 is 1.2 nearly.

$$\frac{63-\mu}{\sigma} = 1.2$$

$$1.2\sigma + \mu = 63 \quad \text{——} (2)$$

Solve (1) & (2)

$$-1.47\sigma + \mu = 35$$
$$1.20\sigma + \mu = 63$$
$$(-) \quad (-) \quad (-)$$

$$-1.67\sigma = 28$$

$$\sigma = \frac{28}{-1.67}$$

$$\sigma = 10.48$$

$$\mu = 50.41.$$

3. The marks obtained by a number of students for a certain subject is assumed to be normally distributed with mean 65 and standard deviation 5. if 3 students are taken at random from this set, what is the probability that exactly ~~two~~ one of them will have marks over ~~70~~ 75?

Let 'x' be the random variable which denotes the marks obtained by students.

Given:

$\mu = 65$

$\sigma = 5$

The standard normal variation is

$$Z = \frac{x-\mu}{\sigma} = \frac{x-65}{5}$$

To find $P(x > 70)$

When $x = 70$

$$Z = \frac{70-65}{5} = \frac{5}{5} = 1$$

$$P(x > 70) = P(Z > 1)$$

$$= 0.5 - P(0 < Z < 1)$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

$P(\text{a student score} > 70) = 0.1587$

$p = 0.1587$

$q = 1 - 0.1587 = 0.8413$

$n = 8$

$P(X = x) = nCx\ p^{x} q^{n-x}$

1

Solution:

Let 'X' be the random Variable denoting the life time of a light bulb.

Given

$\mu = 800$

$\sigma = 40$

$Z = \dfrac{X-\mu}{\sigma}$

$= \dfrac{X-800}{40}$

(i) P(a bulb burns more than 834 hrs)

$= P(X > 834)$

When $X = 834$ $Z \Rightarrow = \dfrac{834-800}{40} = \dfrac{34}{40} = 0.85$

$P(X>834) = P(Z>0.85)$

$= 0.5 - P(0 < Z < 0.85)$

$= 0.5 - 0.3023$

$= 0.1977$

(ii) $P(778 < X < 834)$

When $X = 778 \Rightarrow Z = \dfrac{778-800}{40} = -0.55$

When $X = 834 \Rightarrow Z = 0.85$

$P(778 < X < 834) = P(-0.55 < Z < 0.85)$