CHAPTER–III

RANDOM PROCESS

3.1 INTRODUCTION

Random signals are encountered in every practical communication system. Random signals, unlike deterministic signals are unpredictable. For example, the received signal in the receiver consists of message signal from the source, noise and interference introduced in the channel. All these signals are random in nature, i.e. these signals are uncertain or unpredictable. These kind of random signals cannot be analyzed using Fourier transform, as they can deal only deterministic signals. The branch of mathematics which deals with the statistical characterization of random signals is probability theory.

3.1.1 Basic Terms in Probability Theory

- **Experiment:** Any activity with an observable result is called as experiment. Examples are rolling a die, tossing a coin or choosing a card.
- **Outcome:** The result of an experiment is called as outcome.
- **Sample space:** The set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S. For example, in tossing a coin experiment, the sample space S = {Head, Tail}
- **Event:** Any subset of the sample space is known as an event. For example, in tossing a coin experiment, E = {Head} is the event that a head appears (or) E = {Tail} is the event that a tail appears.
- **Independent event:** Two events are said to be independent, when the occurrence of one event is not affected by the occurrence of other event. If A and B are independent events, then

$$\mathbf{P(A \cap B) = P(A) * P(B)}$$

- **Mutually exclusive events:** Two events are said to be mutually exclusive or disjoint events, when they cannot occur simultaneously (None of the outcomes are common). If A and B are mutually exclusive events, then

$$\mathbf{P(A \cap B) = 0}$$

- **Probability:** It is the measure of possibility or chance that an event will occur. It is defined as:

$$\textbf{Probability} = \frac{\textbf{Number of desirable outcome}}{\textbf{Total number of possible outcomes}} = \frac{\mathbf{E}}{\mathbf{S}}$$

For example, in coin tossing experiment the probability of getting "Head" is 1/2.

- **Axioms of Probability:**

(a) Probability of sample space is 1, i.e. $\mathbf{P(S)=1}$

(b) Probability of an event P(E) is nonnegative real number ranges from 0 to 1.

$$\mathbf{0 \le P(E) \le 1}$$

(c) Probability of the mutually exclusive events is equal to sum of their individual probability.

$$\mathbf{P(A \cup B) = P(A) + P(B)}$$

If A and B are non-mutually exclusive events, then

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

- **Conditional Probability:** The conditional probability of an event B is the probability that the event will occur given that an event A has already occurred. It is defined as,

$$\mathbf{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

Similarly, the conditional probability of event A given that an event B has already occurred is given by,

$$\mathbf{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

If the event **A *and* B *are independent***, then $\mathbf{P(A \cap B) = P(A) * P(B)}$**,** then the Conditional probability **of B given A** is simply P(B) as,

$$\mathbf{P(B/A) = [\,P(A) * P(B)] / P(A) = P(B)}$$

Similarly, conditional probability of A given B is simply P(A) as,

$$\mathbf{P(A/B) = [\,P(A) * P(B)] / P(B) = P(A)}$$

- **Law of Total Probability:** If the events $A_1, A_2, \ldots. A_k$ be mutually exclusive, then the probability of other event B is given by,

$$\mathbf{P(B) = \sum_{\boldsymbol{i}=1}^{\boldsymbol{k}} P(B/A_{\boldsymbol{i}})\,P(A_{\boldsymbol{i}})}$$

- **Baye's Theorem:** Baye's theorem or Baye's rule describes the probability of an event based on the prior knowledge of the related event. It is defined as:

$$P(A_i / B) = \frac{P(B/A_i)\,P(A_i)}{P(B)} = \frac{P(B/A_i)\,P(A_i)}{\sum_{i=1}^{k} P(B/A_i)\,P(A_i)}$$

where,

$P(A_i)$	–	Prior probability or Marginal probability of A, as it does not takes into account any information about B.
P(B)	–	Prior probability or marginal probability of B,as it does not takes into account any information about A.
$P(A_i/B)$	–	Conditional probability of A given B. It is also called as posterior probability, as it depends on the value of B
$P(B/A_i)$	–	Conditional probability of B given A

3.2 RANDOM VARIABLES

The outcomes of a random experiment are not the convenient representation for mathematical analysis. For example, 'Head' and 'Tail' in tossing the coin experiment.It will be convenient, if we assign a number or a range of values to the outcomes of a random experiment. For example, a 'Head' corresponds to 1 and a 'Tail'corresponds to 0.

A random variable is defined as a process of mapping the sample space Q to the set of real numbers. In other words, a random variable is an assignment of real numbers to the outcomes of a random experiment as shown in Figure 3.1.Random variables are denoted by capital letters, i.e., X, Y, and so on, and individual values of the random variable X are X(*w*).

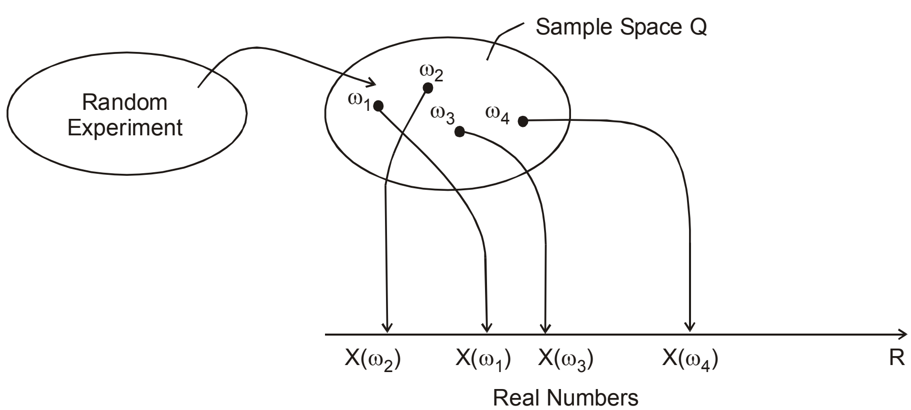


Figure 3.1 Random variable – Mapping from Q to R

For example, consider an experiment of tossing three fair coins. Let, X denote the number of "Heads" that appear, then X is a random variable taking on one of the values 0,1,2,3 with respective probabilities:

$$P\{X = 0\} = P\{(T, T, T)\} = 1/8$$

$$P\{X = 1\} = P\{(T, T, H), (T, H, T), (H, T, T)\} = 3/8$$

$$P\{X = 2\} = P\{(T, H, H), (H, T, H), (H, H, T)\} = 3/8$$

$$P\{X = 3\} = P\{(H, H, H)\} = 1/8$$

3.2.1 Classification of Random Variables

Random variables are classified into continuous and discrete random variables.

- The values of ***continuous random variable*** are continuous in a given continuous sample space. A continuous sample space has infinite range of values. The discrete value of a continuous random variable is a value at one instant of time. For example the Temperature, T at some area is a continuous random variable that always exists in the range say, from T_1 and T_2.

  Examples of continuous random variable – Uniform, Exponential, Gamma and Normal.

- The values of a ***discrete random variable*** are only the discrete values in a given sample space. The sample space for a discrete random variable can be continuous, discrete or even both continuous and discrete points. They may be also finite or infinite. For example the "Wheel of chance" has the continuous sample space. If we define a discrete random variable *n* as integer numbers from 0 to 12, then the discrete random variable is X = {0,1,3,4…..12}.

  Examples of discrete random variable – Bernoulli, Binomial, Geometric and Poisson.

3.2.2 Distribution and Density Function

The **Cumulative Distribution Function (CDF)** of a random variable X is defined as,

$$\mathbf{F_x(x) = P\{\omega \in \Omega : X(w) \leq x\}}$$

which can be simply written as, $\mathbf{F_x(x) = P(X \leq x)}$

Properties of CDF:

- $F_X(x)$ is a non-decreasing function of x.
- Since CDF is a probability, it ranges from 0 to 1,i.e., $0 \le F_X(x) \le 1$
- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$
- $F_X(x)$ is continuous from the right
- $P(a < X \le b) = F_X(b) - F_X(a)$
- $P(X = a) = F_X(a) - F_X(a^-)$

The **Probability Density Function (PDF)**, of a continuous random variable X is defined as the derivative of its CDF. It is denoted by:

i.e.,
$$\boldsymbol{f_X(x) = \frac{d}{dx} F_X(x).}$$

Properties of PDF:

- $f_X(x) \ge 0$
- $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$
- $\int_a^b f_X(x)\,dx = P(a < X \le b)$
- $P(X \in A) = \int_A f_X(x)\,dx$
- $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du.$

The **Probability Mass Function (PMF),** of a discrete random variable X is defined as $p_x(x)$, where

$$\boldsymbol{p_x(x) = P(X = x)}$$

Properties of PMF:

- $0 \le p_x(x_i) \le 1,\ i = 1, 2, ...$
- $p_x(x) = 0, \quad \text{if } x \ne x_i\ (i = 1, 2, ...)$
- $\sum_i p_x(x_i) = 1.$

3.2.3 Statistical Averages of Random Variables

Even though the distribution function provides a complete description of the random variable, some other statistical averages such as mean and variance are also used to describe the random variable in detail.

Mean or Expectation of the random variable: The mean of the continuous random variable X with a density function of $f_X(x)$ is defined as:

$$\mathbf{E[X]=\int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

The mean of the discrete random variable X is defined as the weighted sum of the possible outcomes given as:

$$\mathbf{\mu_x = E[X]=\sum_X x\, P[X=x]}$$

For example, if X is considered as a random variable representing the observations of the voltage of a random signal, then the ***mean value represents the average voltage or dc offset of the signal***.

Variance of the Random Variable: It provides an estimate of the spread of the distribution about the mean. The variance of the continuous random variables is defined as:

$$\boldsymbol{\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2\, f_X(x)\, dx}$$

The variance of the discrete random variables is given by the expectation of the squared distance of each outcome from the mean value. It is defined as:

$$\sigma_x^2 = \mathrm{Var(X)} = \mathrm{E}[(\mathrm{X}-\mu_x)^2]$$

We know that, expectation of a variable is

$$\mathrm{E}[\mathrm{X}]=\sum_{\mathrm{X}} x\, \mathrm{P}[\mathrm{X}=x]$$

Therefore, the variance is given by,

$$\boldsymbol{\sigma_x^2 = \sum_X (x-\mu_x)^2\, P[X=x]}$$

For example, if X is considered as a random variable representing observations of the voltage of a random signal, then the ***variance represents the AC power of the signal***. The second moment of X, **E[X^2]** ***is also called the mean-square value of the random signal and it represents the total power of the signal***.

Covariance of the Random Variable: In communication system, it is very important to compare two signals in order to extract the information. For example, RADAR system compares the transmitted signal to the target with the received (reflected) signal from the target to measure the parameters like range, angle and speed of the object. Correlation and covariance are mainly used for this comparison purpose.

The covariance of two random variables X and Y is defined as the expectation of the product of the two random variables given by:

$$\mathbf{Cov(X, Y) = E[(X - \mathit{m}_X)(Y - \mathit{m}_Y)]}$$

Expanding and simplifying above equation we get:

$$\mathbf{Cov(X, Y) = E[XY] - \mu_X \, \mu_Y}$$

If Cov(X,Y) = 0, then X and Y are uncorrelated, i.e., $\mathbf{E[XY] = \mu_X \, \mu_Y = E[X]E[Y]}$

If X and Y are independent, then **Cov(X, Y) = 0*****, i.e., X and Y are uncorrelated but the converse is not true.***

The correlation coefficient of two random variable is defined as,

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

where, the correlation coefficient ranges from [– 1, 1].

Some of the important random variables used in communication systems are,

(a) Binomial Random Variable:

This is a discrete random variable. Suppose there are 'n' independent trials, each of which results in a success and failure with probability p and $1 - p$ respectively. If X represents the number of successes that occur in the n trials, then X is said to be binomial random variable with parameters (n, p).

The probability mass function of a binomial random variable with parameters n and p is given by,

$$\mathbf{P(X = \mathit{i}) = \binom{n}{i} \mathit{p}^{i} (1 - \mathit{p})^{n-i}, \qquad 0 \le \mathit{i} \le \mathit{n}}$$

The mean and variance of binomial random variable is given by,

$$\mu = \mathbf{E(X)} = np$$

$$\sigma^2 = np(1 - p)$$

Application:

Binomial Random Variable can be used to model the total number of bits received in error when sequence of n bits is transmitted over a channel with a bit-error probability of p as shown in Figure 3.2.

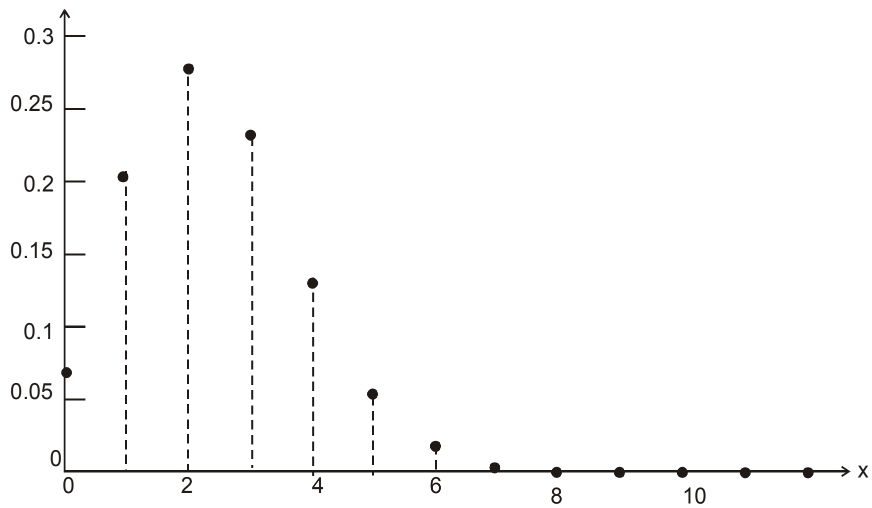


Figure 3.2 PMF for Binomial Random Variable

(b) Uniform Random Variable:

This is a continuous random variable that takes values between a and b with equal probabilities for intervals of equal length. The probability density function is shown in Figure 3.3 defined as:

$$f_X(x)=\begin{cases} \dfrac{1}{b-a}, & a<x<b \\ 0, & \text{Otherwise} \end{cases}$$

The cumulative distribution function is shown in Figure 3.4 defined as:

$$F_X(x)=\begin{cases} 0, & x<a \\ \dfrac{x-a}{b-a}, & a\le x\le b \\ 1, & x>b \end{cases}$$

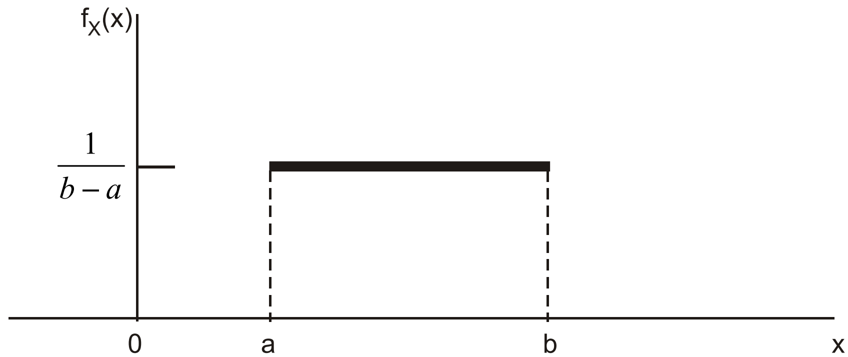


Figure 3.3 PDF for uniform random variable

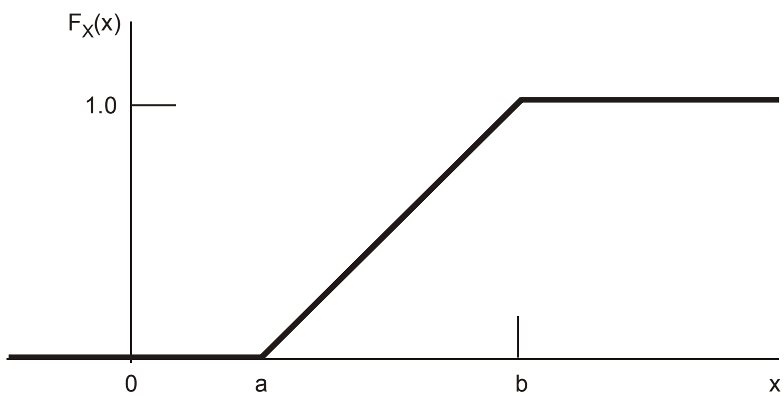


Figure 3.4 CDF for uniform random variable

Application:

Uniform random variable is used to model continuous random variables, whose range is known, but other information like likelihood of the values that the random variable can assume is unknown. For example, the ***phase of a received sinusoid carrier*** is usually modelled as a uniform random variable between 0 and 2π.***Quantization Errors*** are also modelled as uniform random variable.

(c) Gaussian or Normal Random Variable:

The Gaussian or Normal Random Variable is a continuous random variable described by the probability density function as:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

The PDF of Gaussian random variable is a bell shaped that is symmetric about the mean m and attains the maximum value of $\mathbf{1/\sqrt{2\pi}\,\sigma}$ at $x = m$, as shown in Figure 3.5.

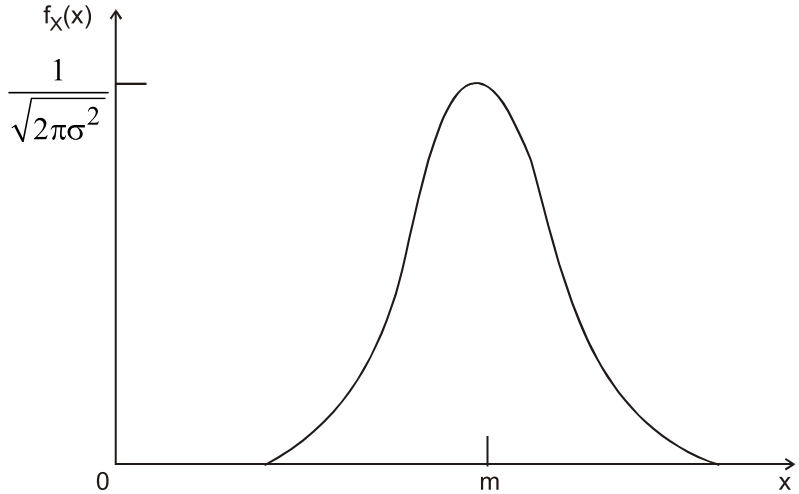


Figure 3.5 PDF for Gaussian Random Variable

Theparameter m is called the ***Mean*** and can assume any finite value. The parameter σ is calledthe ***Standard Deviation*** and can assume any finite and positive value. The square of the standard deviation, i.e., σ^2 is the ***Variance***. A Gaussian random variable with mean m and variance σ^2 is denoted by $N(m, \sigma^2)$. The random variable N(0, 1) is usually called ***standard normal***.

Properties of Gaussian Random Variable:

- It is completely characterized by its mean and variance.
- The sum of two independent Gaussian random variables is also a Gaussian random variable.
- The weighted sum of N independent Gaussian random variables is a Gaussian random variable.
- If two Gaussian random variables have zero covariance (uncorrelated), they are also independent.
- A Gaussian random variable plus a constant is another Gaussian random variable with the mean adjusted by the constant.
- A Gaussian random variable multiplied by a constant is another Gaussian random variable where both the mean and variance are affected by the constant.

Applications:

- The Gaussian random variable is the most important and frequently encountered random variable in communication systems. The reason is that ***thermal noise, which is the major source of noise in communication, has a Gaussian distribution***.
- In Robotics, Gaussian PDF is used to statistically characterize sensor measurements, robot locations and map representations.

3.3 CENTRAL LIMIT THEOREM

An important result in probability theory that is closely related to the Gaussian distribution is the ***Central Limit Theorem (or) Law of large numbers***. Let $X_1, X_2, X_3, \ldots, X_n$ be a set of random variables with thefollowing properties:

- The X_k with $k = 1,2,\ldots n$ are statistically independent.
- The X_k all have the same probability density function.
- Both the mean and the variance exist for each X_k.

We do not assume that the density function of the X_k is Gaussian. Let Y be a new random variable defined as:

$$\mathbf{Y} = \sum_{k=1}^{n} \mathbf{X}_k$$

Then, according to the central limit theorem, the normalized random variable,

$$\mathbf{Z} = \frac{\mathbf{Y} - \mathbf{E[Y]}}{\sigma_{\mathbf{Y}}}$$

approaches a Gaussian random variable with zero mean and unit variance as the number of the random variables $X_1, X_2, X_3, \ldots, X_n$ increases without limit. That is, as n becomes large,the distribution of Z approaches that of a zero-mean Gaussian random variable with unit variance, as shown by:

$$F_Z(z) \to \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{s^2}{2}\right\} ds$$

This is a mathematical statement of the central limit theorem. In words, ***the normalized distribution of the sum of independent, identically distributed random variables***

approaches a Gaussian distribution as the number of random variables increases, regardless of the individual distributions. Thus, Gaussian random variables are common because they characterize the asymptotic properties of many other types of random variables.

When n is finite, the Gaussian approximation is most accurate in the central portion of the density function (hence the name central limit) and less accurate in the "tails" of the density function.

Applications of Central Limit Theorem:

- Channel Modelling
- Finance
- Population Statistics
- Hypothesis Testing
- Engineering Research

3.4 Random Process

A random process or stochastic process is the natural extension of random variables when dealing with signals. In analyzing communication systems, we basically deal with time-varying signals. So far, the assumption is that, all the signals are deterministic. In many situations, it is more appropriate to model signals as ***random rather than deterministic functions***.

One such example is the case of **thermal noise** in electronic circuits. This type of noise is due to the random movement of electrons as a result of thermal agitation, therefore, the resulting current and voltage can only be described statistically. Another situation where modeling by random processes proves useful is in the characterizationof information sources. **An information source**, such as a speech source, generates time-varying signals whose contents are not known in advance. Otherwise there would be no need to transmit them. Therefore, random processes also provide a natural way to model information sources.

A random process is a collection (or ensemble) of random variables $\{\mathbf{X}(t,\omega)\}$ that are functions of a real variable, namely time t where $\omega \in \mathbf{S}$ (Sample space) and $t \in \mathbf{T}$ (Parameter set or Index set).The set of possible values of any individual member of the random process is called state space. Any individual member itself is called a sample function or a realization of the process. A random process can be viewed as a mapping of sample space S to the set of signal waveforms as shown in Figure 3.6.

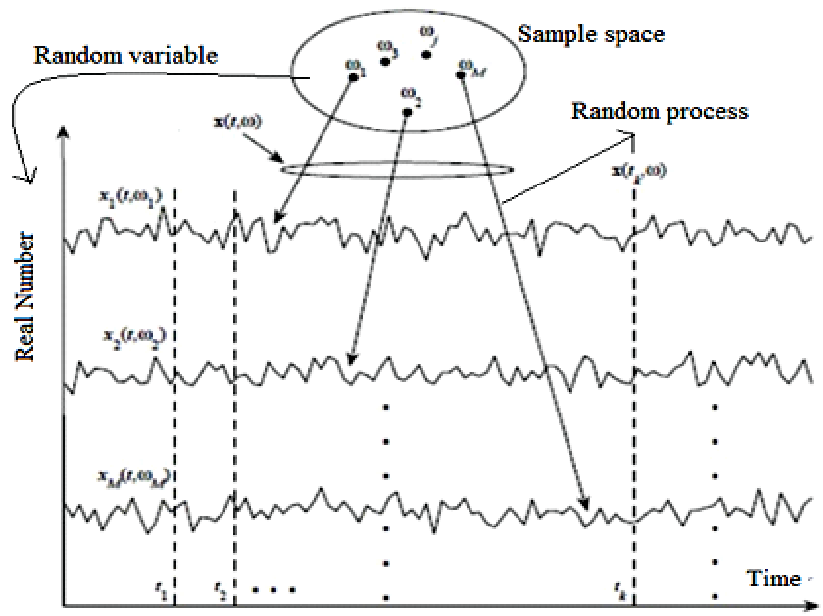


Figure 3.6 Random Process – Mapping of Sample Space to Signal Waveform

The realization of one from the set of possible signals is governed by some probabilistic law. This is similar to the definition of random variables where one from a set of possible values is realized according to some probabilistic law. ***The difference is that in random processes, we have signals (function of time) instead of values (numbers).***

3.4.1 Classification of Random Process

Based on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

- ***If both T and S are discrete, the random process is called a Discrete random sequence.***

  For example, if X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n,\ n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \cdots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

- ***If* T *is discrete and S is continuous, the random process is called a continuous random sequence.***

  For example, if X_n represents the temperature at the end of the n^{th} hourof a day, then $\{X_n, 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

- ***If T is continuous and S is discrete, the random process is called a Discrete random process***.

  For example, if X(t) represents the number of telephone calls received in the interval (0, t) then {X(t)} is a discrete random process, since S = {0, 1, 2, 3, · · ·}.

- ***If both* T *and* S *are continuous, the random process is called a continuous random process.***

  For example, if X(t) represents the maximum temperature at a place in the interval (0, t), {X(t)} is a continuous random process.

  Based on the stationarity, a random process can be classified into stationary and non-stationary random process as shown in Figure 3.7.

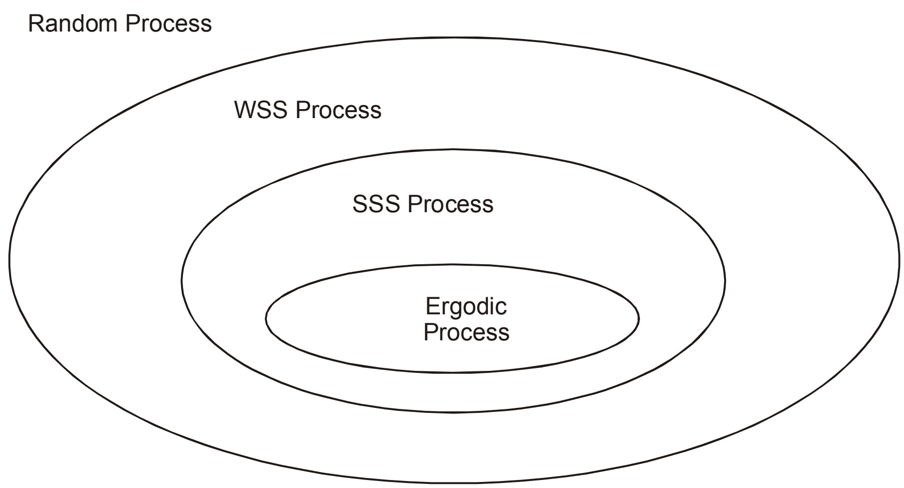


Figure 3.7 Hierarchical Classification of Random Process

- A random process whose ***statistical characteristics do not change with time*** is called as ***Stationary Random Process or Stationary Process***.Example – Noise process as its statistics do not change with time.
- A random process whose ***statistical characteristics changes with time*** is called as ***non-stationary process***. Example – Temperature of a city as temperature statistics depend on the time of the day.

3.5 STATIONARY PROCESS

The random process X(t) is said to be ***Stationary in the Strict Sense* (SSS) *or strictly stationary*** if the following condition holds:

$$F_{X(t_1+\tau),\ldots\ldots,x(t_k+\tau)}(x_1,\ldots\ldots,x_k)=F_{X(t_1),\ldots\ldots,x(t_k)}(x_1,\ldots\ldots,x_k)$$

For all time shifts τ, all k and all possible choices of observation times $t_1,\ldots\ldots.t_k$. In other words, if the joint distribution of any set of random variables obtained by observing the random process X(t) **is invariant** with respect to the location at the time origin $t = 0$.

3.5.1 Mean of the Random Process

The mean of process X(t) is defined as the expectation of the random variable obtained by observing the process at some time t given by:

$$\boldsymbol{m_{\mathrm{X}}(t) = \mathrm{E}[\mathrm{X}(t)] = \int_{-\infty}^{\infty} x\, f_{\mathrm{X}(t)}(x)\, dx}$$

where, $f_{\mathrm{X}(t)}(x)$ is the first order probability density function of the process.

The ***mean of the strict stationary process is always constant*** given by:

$$\boldsymbol{m_x(t) = m_x} \text{ for all values of } t$$

3.5.2 Correlation of the Random Process

Autocorrelation of the process X(t) is given by the expectation of the product of two random variables X(t_1) and X(t_2) obtained by observing the process X(t) at times t_1 and t_2 respectively. ***It is defined as the measure of similarity of random processses.***

$$\mathrm{R_{XX}}(t_1, t_2) = \mathrm{E}[\mathrm{X}(t_1)\,\mathrm{X}(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{X}_1\,\mathrm{X}_2\, f_{\mathrm{X}(t_1),\mathrm{X}(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

For a stationary random process $f_{\mathrm{X}(t_1),\mathrm{X}(t_2)}(x_1, x_2)$ depends only on the difference between the observation time t_1 and t_2. This implies the ***autocorrelation function of a strictly stationary process depends only on the time difference $t_2 - t_1$*** given by:

$$\mathbf{R_{XX}}(\boldsymbol{t_1, t_2}) = \mathbf{R_{XX}}(\boldsymbol{t_2 - t_1}) \text{ for all values of } t_1 \text{ and } t_2$$

3.5.2.1 Importance of Autocorrelation

Autocorrelation function provides the spectral information of the random process. The frequency content of a process depends on the rapidity of the amplitude change with time. This can be measured by correlating amplitudes at t_1 and $t_1 + \tau$. Autocorrelation function can be used to provide information about the rapidity of amplitude variation with time which in turn gives information about their spectral content.

For example, consider two random process $x(t)$ and $y(t)$, whose autocorrelation function $R_X(\tau)$ and $R_Y(\tau)$as shown in Figure 3.8. We can observe from their autocorrelation function that the random process $x(t)$ is a slowly varying process compared to the process $y(t)$. In fact the power spectral density of random process is obtained from the Fourier Transform of their autocorrelation function.

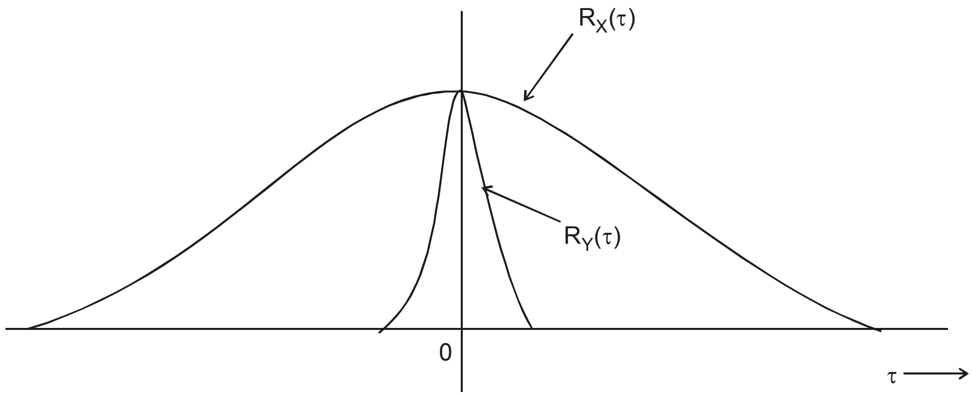


Figure 3.8 Autocorrelation Function of Two Random Process x(t) and y(t)

3.5.2.2 Properties of Autocorrelation Function

- The mean square value of a random process is equal to the value of autocorrelation at $\tau = 0$.

$$\mathbf{R_{XX}(0) = E[X^2(T)]}$$

- The autocorrelation function is an even function of τ.

$$\mathbf{R_{XX}(\tau) = R_{XX}(-\tau)}$$

- The autocorrelation function is maximum at $\tau = 0$.

$$\mathbf{|R_{XX}(\tau)| \leq R_{XX}(0)}$$

- If E[X(t)] $\neq$ 0 and X(t) is ergodic with no periodic components, then

$$\lim_{|\tau| \to \infty} \mathbf{R_{XX}(\tau) = \overline{X}^2}$$

- If X(t) has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
- If X(t) is ergodic, zero mean and has no periodic components, then

$$\lim_{|\tau| \to \infty} \mathbf{R_{XX}(\tau) = 0}$$

3.5.3 Covariance of the Random Process

The autocovariance function of X(t) is defined as:

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - m_X(t_1)][X(t_2) - m_X(t_2)]\}$$
$$= R_{XX}(t_1, t_2) - m_X(t_1)\, m_X(t_2)$$
$$\mathbf{C_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1) - m_X^2}$$

Like autocorrelation function, the autocovariance function of a strictly stationary process depends on the time difference $t_2 - t_1$.

3.6 WHITE NOISE PROCESS

One of the very important random process is the white noise process. Noises in many practical situations are approximated by the white noise process. Most importantly, the white noise plays an important role in modeling WSS signals.

A random process is said to be white noise process X(t), if it is zero mean and power spectral density is defined as:

$$\mathbf{S_X(\omega) = N_0 / 2,} \qquad \textbf{for all frequencies}$$

where, N_0 is a real constant.

The corresponding autocorrelation function is given by:

$$\boxed{\mathbf{R_X(\tau) = (N_0 / 2)\,\delta(\tau)}}$$

where, $\delta(\tau)$ is the Dirac delta function. The PSD and autocorrelation function of a white noise is shown in Figure 3.9 (a) and (b) respectively.

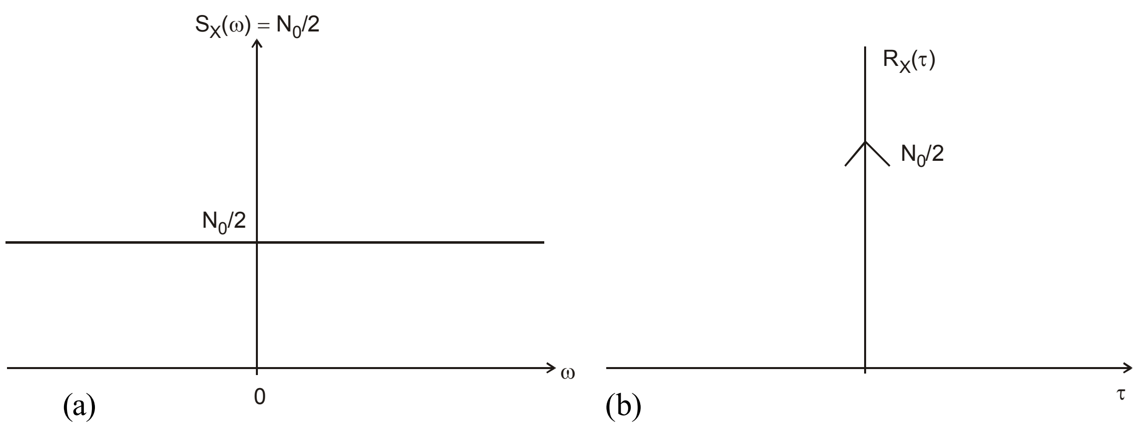


Figure 3.9 (a) PSD of White Noise (b) Autocorrelation Function of White Noise

3.6.1 Properties of White Noise Process

- The term white noise is analogous to white light which contains all visible light frequencies.
- A white noise is a mathematical abstraction; it cannot be physically realized since it has infinite average power.
- A white noise process can have any probability density function.
- The random process X(t) is called as white Gaussian noise process, if X(t) is a stationary Gaussian random process with zero mean and flat power spectral density.
- If the system bandwidth (BW) is sufficiently narrower than the noise BW and noise PSD is flat, we can model it as a white noise process. Thermal noise, which is the noise generated in resistors due to random motion electrons, is generally modelled as white Gaussian noise, since they have very flat PSD over very wide band of frequency.
- A white noise process is called **strict-sense white noise process**, if the noise samples at distinct instants of time are independent.

3.7 WIDE-SENSE STATIONARY PROCESS (WSS)

A process may not be stationary in the strict sense, still it may have mean value $m_x(t)$ and an autocorrelation function which are independent of the shift of time origin.

Ø $m_x(t) = \text{constant}$

Ø $R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1)$

Such a process is known as ***Wide-Sense Stationary or Weakly Stationary Process (WSS) or Co-Variance Stationary***.

3.8 POWER SPECTRAL DENSITY (PSD)

A random process is a collection of signals and the spectral characteristics of these signals determine the spectral characteristics of the random process.

If the signals of the random process are:

Slowly Varying: Random process will mainly contain low frequencies and its power will be mostly concentrated at **low frequencies**.

Fast Varying: Most of the power in the random process will be at the **high-frequency** components.

A useful function that determines the distribution of the power of the random process at different frequencies is the ***Power Spectral Density or Power Spectrum of the random process***, the power spectral density of a random process X (t) is denoted by $S_X(f)$, and denotes the strength of the power in the random process as a function of frequency. The unit for power spectral density is **watts per Hertz (W/Hz)**.

3.8.1 Expression for power spectral density

The impulse response of a linear time invariant filter is equal to the inverse Fourier Transform of the frequency response of the system. Let H(f) denotes the frequency response of the system, thus

$$h(\tau_1) = \int_{-\infty}^{\infty} \mathrm{H}(f) \exp(j2\pi f \tau_1)\, df$$

Substituting $h(\tau_1)$ in $\mathrm{E}[\mathrm{Y}^2(t)]$, we get

$$\mathrm{E}[\mathrm{Y}^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \mathrm{H}(f)\exp(j2\pi f\tau_1)\,df\right] h(\tau_2)\,\mathrm{R}_{XX}(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

$$= \int_{-\infty}^{\infty} df\, \mathrm{H}(f) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2) \int_{-\infty}^{\infty} \mathrm{R}_{XX}(\tau_2-\tau_1)\exp(j2\pi f\tau_1)\, d\tau_1$$

In the last integral on the right hand side of above equation a new variable is defined :

$$\tau = \tau_2 - \tau_1$$

Then the above equation can be rewritten as:

$$\mathrm{E}[\mathrm{Y}^2(t)] = \int_{-\infty}^{\infty} df\, \mathrm{H}(f) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\exp(j2\pi f\tau_2) \int_{-\infty}^{\infty} \mathrm{R}_{XX}(\tau)\exp(-j2\pi f\tau_1)\, d\tau_1$$

The middle integral corresponds to $\mathrm{H}^*(f)$, the complex conjugate of the frequency response of the filter and so we may simplify the equation as:

$$\mathrm{E}[\mathrm{Y}^2(t)] = \int_{-\infty}^{\infty} df\, |\mathrm{H}(f)|^2 \int_{-\infty}^{\infty} \mathrm{R}_{XX}(\tau)\exp(-j2\pi f\tau_1)\, d\tau_1$$

where $|\mathrm{H}(f)|$ is the magnitude response of the filter. In the above equation the last integral term is the Fourier transform of the autocorrelation function of the input random process X(t).This provides the definition of a new parameter,

$$\mathbf{S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-j2\pi f\tau_1)\, d\tau_1}$$

This function is called $S_{XX}(f)$ is called the power spectral density or power spectrum of the stationary process X(t).

Finally, $\mathbf{E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_{XX}(f)\, df}$

The mean square of the output of a stable linear time-invariant filter in response to a stationary process is equal to the integral over all frequencies of the power spectral density of the input process multiplied by the squared magnitude response of the filter.

3.8.2 Relationship between power spectral density and autocorrelation

The power spectral density and the autocorrelation function of a stationary process form a Fourier-transform pair with τ and f as the variables of interest,

$$\mathbf{S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-j2\pi f\tau)\, d\tau}$$

$$\mathbf{R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f) \exp(j2\pi f\tau)\, df}$$

The above relations provide insight of spectral analysis of random processes and they together called as **Einstein-Wiener-Khintchine Relation.**

3.8.3 Properties of Power Spectral Density

1. The zero-frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function (substituting $f = 0$ in $S_{xx}(f)$).

$$\mathbf{S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau}$$

2. The mean square value of a stationary process equals the total area under the graph of the power spectral density (substituting $\tau = 0$ in $R_{xx}(\tau)$)

$$\mathbf{E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(f)\, df}$$

3. The power spectral density of a stationary process is always nonnegative,

$$\mathbf{S_{XX}(f) \leq 0 \quad \text{for all } f}$$

4. The power spectral density of a real valued random process is an even function of frequency, that is,

$$\mathbf{S_{XX}(-f) = S_{XX}(f)}$$

5. The power spectral density appropriately normalized has the properties usually associated with a probability density function:

$$P_X(f) = \frac{S_{XX}(f)}{\int_{-\infty}^{\infty} S_{XX}(f)\,df}$$

3.9 ERGODIC PROCESS

The expectations or ensemble averages of a random process are averages **across the process** that describes all possible values of the sample functions of the process observed at time t_k. Time averages are defined as long term sample averages that are averages **along the process**.

Time averages are the practical means for the estimation of ensemble averages of the random process. The dc value of $x(t)$ is defined by the time average

$$m_x(T) = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

The time average is a random variable as its value depends on the observation interval and sample function. The mean of the time average is given by,

$$E[m_x(T)] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt = m_x$$

The process $x(t)$ is **ergodic in the mean** if,

1. The time average $m_x(T)$ approaches the ensemble average m_x in the limit as the observation interval T approaches infinity, that is

$$\lim_{T\to\infty} m_x(T) = m_x$$

2. The variance of $m_x(T)$ approaches zero in the limit as the observation interval T approaches infinity.

$$\lim_{T\to\infty} \mathrm{var}[m_x(T)] = 0$$

The time averaged autocorrelation function of a sample function $x(t)$

$$R_{XX}(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\,x(t)\,dt$$

The process $x(t)$ is **ergodic in the autocorrelation** function if,

$$\lim_{T\to\infty} R_{XX}(\tau, T)] = R_{XX}(\tau)$$

$$\lim_{T\to\infty} \mathrm{Var}[R_{XX}(\tau, T)] = 0$$

For a random process to be ergodic, it has to be stationary; however the converse is not true.

3.10 GAUSSIAN PROCESS

Gaussian processes play an important role in communication systems. The fundamental reason for their importance is that ***thermal noise in electronic devices, which is produced by the random movement of electrons due to thermal agitation, can be closely modeled by a Gaussian process.***

In a resistor, free electrons move as a result of thermal agitation. The movement of these electrons is random, but their velocity is a functionof the ambient temperature. The higher the temperature, the higher the velocity of the electrons. The movement of these electrons generates a current with a random value. We can consider each electron in motion as a tiny current source, whose current is a random variable that can be positive or negative, depending on the direction of the movement of the electron. The total current generated by all electrons, which is the generated thermal noise, is the sum of the currents of all these current sources. We can assume that at least a majority of these sources behave independently and, therefore, the total current is the sum of a large number of independent and identically distributed random variables. Now, by applying the central limit theorem, we conclude that this total current has a Gaussian distribution. For this reason, thermal noise can be very well modeled by Gaussian random process.

Gaussian processes provide rather good models for some information sources as well. Some properties of the Gaussian processes, make these process mathematically tractable and easy to use.

Let Y be a random variable obtained by integrating the product of a random process $X(t)$ for a time period of $t = 0$ to $t = T$ and some function $g(t)$ given by:

$$Y = \int_0^T X(t)\, g(t)\, dt$$

The weighting function in above equation is such that the mean-square value of the random variable Y is finite and if the random variable Y is a Gaussian distributed random variable for every g(t), then the process X(t) is said to be Gaussian process.

The random variable Y has a Gaussian distribution if its probability density function has the form,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left[-\frac{(y - m_Y)^2}{2\sigma_Y^2}\right]$$

where m_Y is the mean and σ_Y^2 is the variance of the random variable Y.

A plot of this probability density function is shown in Figure 3.10, for the special case when the Gaussian random variable Y is normalized to have mean m_Y of zero and a variance σ_Y^2 of 1.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

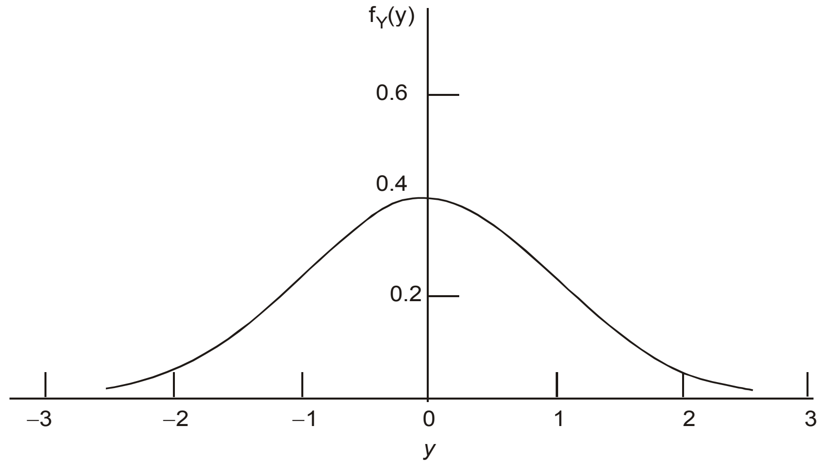


Figure 3.10 Normalized Gaussian Distribution

3.10.1 Advantages of Gaussian Process

1. Gaussian process has many properties that make ***analytic results possible***.
2. Random processes produced by ***physical phenomena are often such that a Gaussian model is appropriate***. Further the use of Gaussian model is confirmed by experiments.

3.10.2 Properties of Gaussian Process

1. If the set of random variables $X(t_1), X(t_2), X(t_3), \ldots \ldots, X(t_n)$ obtained by observing a random process $X(t)$ at times $t_1, t_2, \ldots t_n$. and the process $X(t)$ is Gaussian, then this set of random variables is jointly Gaussian for any n, with their PDF completely specified by the set of means:

$$m_{X(t_i)} = E[X(t_i)], \text{ where } i = 1, 2, 3 \ldots . n$$

and the set of covariance functions,

$$C_X(t_k, t_i) = E[(X(t_k)\ m_{X(t_k)})(X(t_i) - m_{X(t_i)})] \quad k,\ i = 1,2,3 \ldots . n$$

2. If the set of random variables $X(t_i)$ is uncorrelated, that is,

$$C_{ij} = 0 \qquad i \neq j$$

then $X(t_i)$ are independent.

3. If a Gaussian process is stationary, then the process is also strictly stationary.
4. If a Gaussian process X(t) is passed through LTI filter, then the random process Y(t) at the output of the filter is also Gaussian process.

3.11 TRANSMISSION OF A RANDOM PROCESS THROUGH A LTI FILTER

When a random process X(t) is applied as input to a linear time-invariant filter of impulse response $h(t)$, producing a new random process Y(t) at the filter output as shown in Figure 3.11. It is difficult to describe the probability distribution of the Y(t), even when the probability distribution of the X(t) is completely specified.

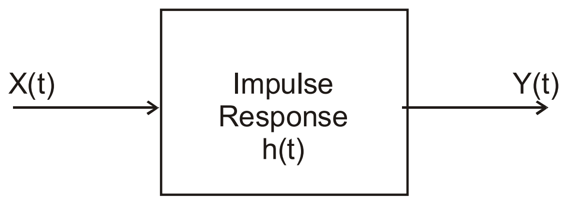


Figure 3.11 Transmission of a random process through a LTI filter

The time domain form of input-output relations of the filter for defining the ***mean and autocorrelation functions*** of the output random process Y(t) in terms of the input X(t), assuming X(t) is a stationary process. The transmission of a process through a LTI filter is governed by the ***convolution integral,*** where the output random process Y(t) is expressed in terms of input random process X(t) as:

$$\mathbf{Y}(t) = \int_{-\infty}^{\infty} h(\tau_1)\,\mathbf{X}(t-\tau_1)\,d\tau_1$$

where, τ_1 is the integration variable.

Mean of Y(t):

$$m_Y(t) = E[Y(t)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\,X(t-\tau_1)\,d\tau_1\right]$$

Provided E[X(t)] is finite for all t and the system is stable.

Interchanging the order of expectation and integration, we get

$$m_Y(t) = \int_{-\infty}^{\infty} h(\tau_1)\, E[X(t-\tau_1)]\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, m_X\,(t-\tau_1)\, d\tau_1$$

When the input random process X(t) is stationary, the mean $m_X(t)$ is a constant m_X, so

$$m_Y = m_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1$$

$$\boldsymbol{m_Y = m_X\, \mathbf{H(0)}}$$

where, H(0) is the zero-frequency (DC) response of the system.

The mean of the output random process Y(t) produced at the output of a LTI system in response to input process X(t) is equal to the mean of X(t) multiplied by the DC response of the system.

Autocorrelation of Y(*t*):

$$R_{YY}(t_1, t_2) = E[Y(t_1)\, Y(t_2)]$$

Using convolution integral, we get

$$R_{YY}(t_1, t_2) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t_1-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_1)\, X(t_2-\tau_2)\, d\tau_2\right]$$

Provided E[X^2(t)] is finite for all t and the system is stable,

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2\, E[X(t_1-\tau_1)\, X(t_2-\tau_2)]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, d\tau_2\, R_{XX}(\tau_1-\tau_1, t_2-\tau_2)$$

When the input X(t) is a stationary process, the autocorrelation function of X(t) is only a function of the difference between the observation times $t_1-\tau_1$ and $t_2-\tau_2$. Thus putting $\tau=\tau_1-\tau_2$ in above equation, we get

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\,h(\tau_2)\,R_{XX}(\tau - \tau_1 + \tau_2)\,d\tau_1\,d\tau_2$$

On combining this result with the mean m_Y ,we see that ***if the input to a stable linear time-invariant filter is a stationary process, then the output of the filter is also a stationary process.***

When $\tau = 0$, $R_{YY}(0) = E[Y^2(t)]$ so

$$\mathbf{E[Y^2(t)]} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{h(\tau_1)\,h(\tau_2)\,R_{XX}(\tau_2 - \tau_1)\,d\tau_1\,d\tau_2}$$

which is a constant.

FORMULAE TO REMEMBER

- If A and B are independent events, then

$$P(A \cap B) = P(A) * P(B)$$

- If A and B are mutually exclusive events, then

$$P(A \cap B) = 0$$

- Conditional probability of event A given B,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

- Law of total probability

$$P(B) = \sum_{i=1}^{k} P(B/A_i)\,P(A_i)$$

- Baye's theorem

$$P(A_i/B) = \frac{P(B/A_i)\,P(A_i)}{P(B)}$$

- Cumulative Distribution Function or CDF of a random variable X

$$F_X(x) = P(X \le x)$$

- Probability Density Function or PDF of a continuous random variable X

$$f_X(x) = \frac{d}{dx} F_X(x)$$

- Probability Mass Function or PMF of a discrete random variable X

$$p_X(x) = P(X = x)$$

- Mean of the random variable

$$E[X] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx \text{ (Continuous)}$$

$$E[X] = \sum_X x\, P[X = x] \text{ (Discrete)}$$

- Variance of the random variable

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_x(x)\, dx \text{ (Continuous)}$$

$$\sigma_X^2 = E[(X - \mu_x)^2] = \sum_x (x - \mu_x)^2 P[X = x] \text{ (Discrete)}$$

- Covariance of two random variable

$$\mathrm{Cov}(X, Y) = E[XY] - \mu_x \mu_y$$

- If Cov(X,Y) = 0, then X and Y are uncorrelated.
- If X and Y are independent, then Cov(X, Y) = 0.
- PMF of Binomial Random Variable

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad 0 \le i \le n$$

- PDF of Uniform Random Variable

$$f_x(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{Otherwise} \end{cases}$$

- PDF of Gaussian or Normal Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

- The random process X(*t*) is said to be Stationary in the Strict Sense (SSS) or Strictly Stationary if the following condition holds,

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots \ldots, x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots \ldots, x_k)$$

- Mean of the random process X(*t*) is defined as the:

$$m_x(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_{x(t)}(x)\, dx$$

- Autocorrelation of the random process X(*t*) is:

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2\, f_{x(t_1), x(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

- Autocovariance function of random variable X(*t*) is:

$$C_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1) - m_x^2$$

- Wide-Sense Stationary or Weakly Stationary Process (WSS) or Co-variance Stationary

$$m_X(t) = \text{constant and } R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1)$$

- Power spectral density or power spectrum of the stationary process X(*t*)

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-j2\pi f \tau_1)\, d\tau_1$$

- Einstein-Wiener-Khintchine relations

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-j2\pi f \tau)\, d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f) \exp(j2\pi f \tau)\, df$$

- *x*(*t*) is ergodic in the mean if

$$\lim_{T \to \infty} m_x(T) = m_x,$$

$$\lim_{T \to \infty} \text{var}[m_x(T)] = 0$$

- *x*(*t*) is ergodic in the autocorrelation function if:

$$\lim_{T \to \infty} R_{XX}(\tau, T) = R_{XX}(\tau),$$

$$\lim_{T \to \infty} \text{Var}[R_{XX}(\tau, T)] = 0$$

- The mean of the output random process Y(t) produced at the output of a LTI system in response to input process X(t) is equal to the mean of X(t) multiplied by the DC response of the system.

$$m_Y = m_X\, H(0)$$

SOLVED EXAMPLES

1. A telegraph source generates two symbols: Dot and Dash. The dots were twice as likely to occur as dashes. What is the probability of occurrence of dot and dash?

Solution: Given that,

$$P(\text{Dot}) = 2\,P(\text{Dash})$$

We know that the probability of sample space is 1, so

$$P(\text{Dot}) + P(\text{Dash}) = 1$$

Substituting, P(Dot) = 2 P(Dash) in above equation, we get

$$3\,P(\text{Dash}) = 1$$

Thus, $$P(\text{Dash}) = 1/3 \text{ and } P(\text{Dot}) = 2/3$$

2. Binary data are transmitted over a noisy communication channel in a block of 16 binary digits. The probability that a received digit is in error due to channel noise is 0.01.Assume that the errors occur in various digit positions within a block are independent. Find a) Mean errors per block, *b*) variance of the number of errors per block and c) Probability that the number of errors per block is greater than or equal to 4.

Solution: Let X denote the random variable of number of errors per block. Then, X has a binomial distribution.

(a) Mean $= np$

$$\text{Given, } n = 16 \text{ and } p = 0.01$$

$$\text{Mean error per block} = 16 * 0.01 = 0.16$$

(b) Variance $= np(1 - p)$

Variance of the number of errors per block = (16)(0.01)(0.99) = 0.158.

(c) Probability that the number of errors per block is greater than or equal to 4 is,

$$P(X \geq 4) = 1 - P(X \leq 3)$$

Using binomial distribution, we get,

$$P(X \leq 3) = \sum_{i=0}^{3} \binom{16}{i} (0.01)^i \, (0.99)^{16-i} = 0.986$$

Hence, $P(X \geq 4) = 1 - 0.986 = 0.014$

3. The PDF of a random variable X is given by $f_X(x) = k$, for $a \le x \le b$ and zero elsewhere, where k is a constant. Find the:

(i) Value of k

(ii) If a = – 1 and b = 2. Calculate P(|X| $\le$ c) for c = 1/2.

Solution:

(i) Given that PDF $f_X(x)$ corresponds to uniform random variable, so the value of k is,

$$k = 1 / (b-a)$$

(ii) Using the value of $k = 1/(2+1) = 1/3$, PDF is given by $f_X(x) = 1/3$, for $-1 \le x \le 2$ and zero elsewhere,

$$P(|X| \le 1/2) = P(-1/2 \le X \le 1/2) = \int_{-1/2}^{1/2} \left(\frac{1}{3}\right) dx$$

We get, $P(|X| \le 1/2) = 1/3.$

4. Find the mean and variance of random variable X that takes the values 0 and 1 with probabilities 0.4 and 0.6 respectively.

Solution:

$$\text{Mean} = E[X] = \sum_X x\, P[X = x] = 0(0.4) + 1(0.6) = 0.6$$

$$\text{Variance} = \sigma_X^2 = \sum_X (x - \mu_X)^2\, P[X = x] = (0 - 0.6)^2 (0.4) + (1 - 0.6)^2 (0.6) = 0.24$$

5. Find the covariance of X and Y if (a) X and Y are independent (b) Y = aX + b.

Solution:

(a) $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$

Since, X and Y are independent, $E[XY] = E[X]\, E[Y]$. $\text{Cov}(X, Y) = 0$

(b) $Y = aX + b$

$$E[XY] = E[X(aX + b)] = a\, E[X^2] + b\, E[X] = a\, E[X^2] + b\, E[X]$$

$$E[Y] = E[aX + b] = aE[X] + E[b] = a\, E[X] + b$$

$$\text{Cov}(X,Y) = a\, E[X^2] + b\, E[X] - E[X]\, (aE[X] + b)$$

$$= a\, E[X^2] + b\, E[X] - aE^2[X] - bE[X]$$

$$\text{Cov}(X,Y) = a\,(E[X^2] - E^2[X])$$

6. Show that the random process $X(t) = A\cos(\omega_c t + \theta)$ is a wide-sense stationary process where, θ is a random variable uniformly distributed in the range $(0, 2\pi)$ and A and ω_c are constant.

Solution: For a random process to be WSS, it is necessary to show that,

- Mean is constant
- Autocorrelation function depends only on time difference.

The PDF of the uniform distribution is given by

$$f(\theta) = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi$$

(a) Mean

$$E[x(t)] = \int_0^{2\pi} X(t)\frac{1}{2\pi}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} A\cos(\omega_c t + \theta)\,d\theta$$

$$= \frac{A}{2\pi}[\sin(\omega_c t + \theta)]_0^{2\pi} = 0$$

Therefore, mean is a constant.

(b) Autocorrelation

$$R_{XX}(t, t+\tau) = E[A\cos(\omega_c t + \theta)\,A\cos(\omega_c t + \omega_c\tau + \theta)]$$

$$= \frac{A^2}{2}\cos(\omega_c\tau) + A^2\,4\pi(0)$$

$$= \frac{A^2}{2}\cos(\omega_c\tau).$$

So, autocorrelation function depends on the time difference τ

As, mean is constant and autocorrelation function depends only on τ, X(t) is a wide-sense stationary process.

7. *Show that the random process X(t) = A cos$(\omega_c t + \theta)$ is a wide-sense stationary* process where, θ and ω_c are constant and A is a random variable.

Solution: For a random process to be WSS, it is necessary to show that,

- Mean is constant
- Autocorrelation function depends only on time difference.

(a) Mean

$$E[X(t)] = E[A\cos(\omega_c t + \theta)]$$
$$= \cos(\omega_c t + \theta)\, E[A]$$
$$E[X(t)] \neq 0$$

Therefore, mean is not a constant.

(b) Autocorrelation

$$R_{XX}(t, t+\tau) = E[A\cos(\omega_c t + \theta)\, A\cos(\omega_c t + \omega_c\tau + \theta)]$$
$$= \frac{1}{2}[\cos(\omega_c \tau) + \cos(2\omega_c t + 2\theta + \omega_c \tau)]\, E[A^2]$$

Thus, the autocorrelation of X(t) is not a function of time difference τ only.So the given random process X(t) is not WSS.

8. Let X(t) = A cos ωt + B sin ωt and Y(t) = B cos t - A sin t, where A and B are independent random variables both having zero mean and variance σ^2, and ω is constant. Find the cross-correlation of X(t) and Y(t).

Solution: The cross-correlation of X(t) and Y(t) is :

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$
$$= E[(A\cos\omega t_1 + B\sin\omega t_1)(B\cos\omega t_2 - A\sin\omega t_2)]$$
$$= E[A\,B](\cos\omega t_1 \cos\omega t_2 - \sin\omega t_1 \sin\omega t_2)$$
$$- E[A^2]\cos\omega t_1 \sin\omega t_2 + E[B^2]\sin\omega t_1 \cos\omega t_2$$

Since, $E[AB] = E[A]E[B] = 0$ and $E[A^2] = E[B^2] = \sigma^2$

$$R_{XY}(t_1, t_2) = \sigma^2(\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2)$$
$$= \sigma^2 \sin\omega(t_1 - t_2)$$
$$R_{XY}(\tau) = -\sigma^2 \sin\omega\tau$$

where, $\tau = t_2 - t_1$.

9. The input X(t) to a diode with a transfer characteristic $Y = X^2$ is a zero mean stationary Gaussian random process with an autocorrelation function $R_{XX}(\tau) = \exp(-|\tau|)$. Find the mean $\mu_Y(t)$ and $R_{YY}(t_1, t_2)$.

Solution:

$$\mu_Y = E[Y(t)] = E[X^2(t)] = R_{XX}(0) = 1$$

$$R_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[X^2(t_1)\, X^2(t_2)]$$

For zero mean Gaussian random variable,

$$E[X^2(t_1)\, X^2(t_2)] = E[X^2(t_1)]\, E[X^2(t_2)] + 2E[X(t_1)X(t_2)]^2$$

where, $$E[X^2(t_1)] = E[X^2(t_2)] = R_{XX}(0)$$

$$E[X(t_1)X(t_2)] = R_{XX}(|t_1 - t_2|)$$

Since X(t) is stationary,

$$R_{YY}(t_1, t_2) = [R_{XX}(0)]^2 + 2\,[R_{XX}(|t_1 - t_2|)]^2$$

or $$R_{YY}(\tau) = [R_{XX}(0)]^2 + 2\,[R_{XX}(\tau)]^2 = 1 + 2\exp(-2|\tau|)$$

10. A wide sense stationary random process X(t) is applied to the input of an LTI system with impulse response $h(t) = 3e^{-2t}\, u(t)$. Calculate the mean of the output Y(t) of the system if E[X(t)] = 2.

Solution: The frequency response of the system can be obtained by taking Fourier transform of the impulse response as:

$$H(\omega) = F[h(t)] = 3\frac{1}{j\omega + 2}$$

The mean value of the output Y(t) can be obtained as:

$$m_Y = m_X\, H(0)$$

$$H(0) = 3*(1/2) = 3/2$$

Therefore, mean of the output $Y(t) = 2 * (3/2) = 3$

11. Let X and Y be real random variables with finite second moments. Prove the Cauchy-Schwarz inequality $(E[XY])^2 \leq E[X^2]\, E[Y^2]$. (April/May 2015)

Solution: The mean square value of a random variable can never be negative value, so

$$E[(X - aY)^2] \geq 0, \quad \text{for any value of } a$$

Expanding above equation we get,

$$E[X^2] - 2a\,E[XY] + a^2\,E[Y^2] \geq 0$$

Substituting $a = E[X\,Y]/E[Y^2]$ to get left hand side of this inequality as minimum, we get

$$E[X^2] - (E[EY]^2 / E[Y^2] \geq 0$$

(or) $$(E[X\,Y])^2 \leq E[X^2]\,E[Y^2]$$

12. Let X(*t*) and Y(*t*) be both zero-mean and WSS random processes. Consider the random process Z(*t*) = X(*t*) + Y(*t*). Determine the autocorrelation and power spectrum of Z(*t*) if X(*t*) and Y(*t*) are jointly WSS. (Apr/May 2015)

Solution: The autocorrelation of Z(t) is given by:

$$R_{ZZ}(t_1, t_2) = E[Z(t_1)Z(t_2)]$$

$$= E[[X(t_1) + Y(t_1)][X(t_2) + Y(t_2)]]$$

$$= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] + E[Y(t_1)X(t_2)] + E[Y(t_1)Y(t_2)]$$

$$= R_{XX}(t_1, t_2) + R_{XY}(t_1, t_2) + R_{YX}(t_1, t_2) + R_{YY}(t_1, t_2)$$

As X(t) and Y(t) are jointly WSS,

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau), \text{ where } \tau = t_2 - t_1$$

We know that the Fourier transform of the autocorrelation function gives power spectrum, taking Fourier transform on both sides, we get

$$S_{ZZ}(\omega) = S_{XX}(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_{YY}(\omega)$$

13. Let $X(t) = A\cos(\omega t + \phi)$ and $Y(t) = A\sin(\omega t + \phi)$ where A and ω are constants and ϕ is a uniform random variable [0,2π]. Find the cross correlation of X(*t*) and Y(*t*). (Apr/May 2015)(May/June 2016)

Solution: The cross-correlation of X(t) and Y(t) is

$$R_{XY}(t, t+\tau) = E[X(t)\,Y(t+\tau)]$$

$$= E[A^2 \cos(\omega t + \phi)\sin(\omega t + \phi)]$$

$$= \frac{A^2}{2} E[\sin(2\omega t + \omega\tau + 2\phi) - \sin(-\omega\tau)]$$

$$= \frac{A^2}{2}\{E[\sin(2\omega t + \omega\tau + 2\phi)] + E[\sin(\omega\tau)]\}$$

$$= \frac{A^2}{2}\{0 + E[\sin(\omega\tau)]\}$$

$$R_{XY}(t, t+\tau) = R_{XY}(\tau) = \frac{A^2}{2}\sin(\omega\tau)$$

14. In a binary communication system, let the probability of sending a 0 and 1 be 0.3 and 0.7 respectively. Let us assume that a 0 being transmitted, the probability of it being received as 1 is 0.01 and the probability of error for a transmission of 1 is 0.1.

(i) What is the probability that the output of this channel is 1?

(ii) If a 1 is received then what is the probability that the input to the channel was 1?

(Nov/Dec 2015)

Solution: Let X and Y denote the input and output of the channel. Given that

$$P(X = 0) = 0.3, \quad P(X = 1) = 0.7$$

$$P(Y = 0 \mid X = 0) = 0.99, \quad P(Y = 1 \mid X = 0) = 0.01$$

$$P(Y = 0 \mid X = 1) = 0.1, \quad P(Y = 1 \mid X = 1) = 0.9$$

(a) We know that, from the total probability theorem.

$$P(Y = 1) = P(Y = 1 \mid X = 0) * P(X = 0) + P(Y = 1 \mid X = 1) * P(X = 1)$$

$$= 0.01 * 0.3 + 0.9 * 0.7$$

$$P(Y = 1) = 0.633$$

The probability that the output of this channel is 1 is 0.633.

(b) Using Baye's rule

$$P(X=1|Y=1) = \frac{P(X=1)\,P(Y=1|X=1)}{P(X=0)\,P(Y=1|X=0) + P(X=1)\,P(Y=1|X=1)}$$

$$= (0.7 * 0.9) / (0.3 * 0.01 + 0.7 * 0.9)$$

$$P(X = 1 \mid Y = 1) = 0.9953$$

If a 1 is received then the probability that the input to the channel was 1 is 0.9953.

15. Given a random process, $X(t) = A\cos(\omega t + \mu)$ where A and ω are constants and μ is a uniformly distributed random variable. Show that X(t) is ergodic in both mean and autocorrelation. (May/June 2016)

Solution: For X(t) to be ergodic in mean and autocorrelation

$$\overline{X} = < X(t) > = E[X(t)] = \mu_x$$

$$\overline{R}_{XX}(\tau) = < X(t)\, X(t+\tau) > = E[X(t)\, X(t+\tau)] = R_{XX}(\tau)$$

Ensemble Average: $E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \mu)\, f_\mu(\mu)\, d\mu$

Assume μ is uniformly distributed over $-\pi$ to π

$$E[X(t)] = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \mu)\, d\mu = 0$$

$$R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \mu)\cos(\omega(t+\tau) + \mu)\, d\mu$$

$$R_{XX}(\tau) = \frac{A^2}{2}\cos\omega\tau$$

Time Average:

$$\overline{X} = < X(t) > = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A\cos(\omega t + \mu)\, dt$$

$$= \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} A\cos(\omega t + \mu)\, dt$$

$$\overline{x} = 0$$

$$\overline{R}_{XX}(\tau) = \langle X(t)\, X(t+\tau)\rangle$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2\cos(\omega t + \mu)\cos(\omega(t+\tau) + \mu)\, dt$$

$$= \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{2}[\cos\omega\tau + \cos(2\omega t + 2\mu + \omega\tau)\, dt$$

$$\overline{R}_{XX}(\tau) = \frac{A^2}{2}\cos\omega\tau.$$

Thus time averaged mean and autocorrelation is equal to ensemble averaged mean and autocorrelation. So the given process X(t) is ergodic in both mean and autocorrelation.

16. Consider two linear filters connected in cascade as shown in Fig.1. Let X(t) be a stationary process with auto correlation function $R_x(\tau)$, the random process appearing at the first input filter is V(t) and the second filter output is Y(t).

(a) Find the autocorrelation function of Y(t).

(b) Find the cross correlation function $R_{vy}(\tau)$ of V(t) and Y(t). (Apr/May 2017)

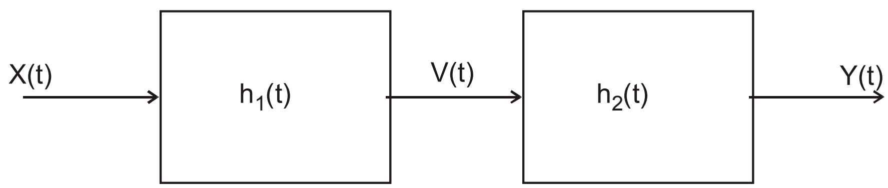


Fig. 1

Solution:

(a) The cascade connection of two filters is equivalent to a filter with the impulse response

$$h(t) = \int_{-\infty}^{\infty} h_1(\tau)\, h_2(t-\tau)\, d\tau$$

The autocorrelation function of Y(t) is given by:

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2$$

(b) The cross correlation of V(t) and Y(t) is given by:

$$R_{VY}(\tau) = E[V(t+\tau)\, Y(t)]$$

The output Y(t) is given as $Y(t) = \int_{-\infty}^{\infty} V(\mu)\, h_2(t-\mu)\, d\mu$

So, $R_{VY}(\tau) = E[V(t+\tau) \int_{-\infty}^{\infty} V(\mu)\, h_2(t-\mu)\, d\mu]$

$$R_{VY}(\tau) = \int_{-\infty}^{\infty} h_2(t-\mu)\, R_V(t+\tau-\mu)\, d\mu$$

17. The amplitude modulated signal is defined as $X_{AM}(t) = A\,m(t)\cos(\omega_c t + \theta)$ where *m*(*t*) is the baseband signal and $A\cos(\omega_c t + \theta)$ is the carrier. The baseband signal *m*(*t*) is modeled as a zero mean stationary random process with the autocorrelation function $R_{xx}(\tau)$ and the PSD $G_x(f)$. The carrier amplitude A and frequency ω_c are assumed to be constant and the initial carrier phase θ is assumed to be random uniformly distributed in the interval $(-\pi, \pi)$. Furthermore, *m*(*t*) and θ are assumed to be independent.

(i) Show that $X_{AM}(t)$ is Wide Sense Stationary

(ii) Find PSD of $X_{AM}(t)$. (Apr/May 2017)

Solution:

(i) For $X_{AM}(t)$ to be WSS, its :

- Mean $E[X_{AM}(t)] = \text{Constant}$
- Autocorrelation $E[X_{AM}(t)\, X_{AM}(t+\tau)]$ depends on τ

$$E[X_{AM}(t)] = E[A\, m(t)\cos(\omega_c t + \theta)]$$

$$= A\, E[m(t)]\, E[\cos(\omega_c t + \theta)]$$

Given that, *m*(*t*) is zero mean stationary random process:

$$E[X_{AM}(t)] = 0 \text{ (Constant)}$$

$$R_{XAMXAM}(\tau) = E[X_{AM}(t)\, X_{AM}(t+\tau)]$$

$$= E[A\, m(t)\cos(\omega_c t + \theta)\, A\, m(t+\tau)\cos(\omega_c (t+\tau) + \theta)]$$

$$= A^2\, E[m(t)\, m(t+\tau)]\, E[\cos(\omega_c (t+\theta)\cos(\omega_c t + \omega_c \tau + \theta)]$$

$$= \frac{A^2}{2} R_{xx}(\tau)\, E[\cos \omega_c \tau + \cos(2\omega_c t + \omega_c \tau + 2\theta)]$$

$$R_{XAMXAM}(\tau) = \frac{A^2}{2} R_{xx}(\tau)\cos \omega_c \tau$$

Since mean of $X_{AM}(t)$ is constant and autocorrelation of $X_{AM}(t)$ depends on τ,

$X_{AM}(t)$ *is Wide Sense Stationary*

(ii) We know that the Fourier transform of the autocorrelation function gives power spectrum.

$$F[R_{XAMXAM}(\tau)] = F[\frac{A^2}{2} R_{xx}(\tau)\cos\omega_c\tau]$$

$$G_{XAMXAM}(\omega) = \frac{A^2}{2} F[R_{xx}(\tau)]\ F[\cos\omega_c\tau]$$

Given that PSD of $m(t)$ is $G_x(f)$.

We know that,

$$F[\cos\omega_c\tau] = \pi\delta(f - f_c) + \pi\delta(f + f_c)$$

Using frequency convolution theorem, we get

$$G_{XAMXAM}(f) = \frac{A^2}{2} G_x(f) * [\pi\delta(f - f_c) + \pi\delta(f + f_c)]$$

$$G_{XAMXAM}(f) = \frac{A^2\pi}{2} [G_x(f - f_c) + G_x(f + f_c)].$$

REVIEW QUESTIONS AND ANSWERS

PART- A

1. Define Random Variable. (Nov/Dec 2015)

A random variable is a function that assigns a real number X(S) to every element $s \in S$, where S is the sample space corresponding to a random experiment E.

Example: Tossing an unbiased coin twice. The outcomes of the experiment are HH, HT, TH,TT. Let X denote the number of heads turning up. Then X has the values 2,1,1,0.Here, X is a random variable which assigns a real number to every outcome of a random experiment.

2. **State Baye's rule. (Nov/Dec 2015)**

Baye's rule or Baye's theorem relates the conditional and marginal probabilities of stochasticevents A and B:

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$$

Where, P(A|B) is the conditional probability of A given B, P(B|A) is the conditional probability of B given A, P(A) is the marginal probability of A and P(B) is the marginal probability of B.

3. Define discrete random variable.

If X is a random variable which can take a finite number or countably infinite number of p values, X is called a discrete RV.Eg. Let X represent the sum of the numbers on the 2 dice, when two dice are thrown.

4. Define continuous random variable.

If X is a random variable which can take all values (i.e., infinite number of values) in an interval, then X is called a continuous RV. Eg. The time taken by a person who speaks over a telephone.

5. Define cumulative distribution function of a random variable.

The Cumulative Distribution Function (CDF) or distribution function of a random variable X is defined as, $F_X(X) = P\{\omega \in \Omega : X(\omega) \le x\}$, which can be simply written as

$$F_x(x) = P(X \le x)$$

6. List the properties of CDF.

1. $0 \le F_x(x) \le 1$
2. $F_x(x)$ is non-decreasing
3. $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to +\infty} F_x(x) = 1$
4. $F_x(x)$ is continuous from the right
5. $P(a < X \le b) = F_x(b) - F_x(a)$
6. $P(X = a) = F_x(a) - F_x(a^-)$.

7. Define probability density function of a random variable.

The Probability Density Function or PDF of a continuous random variable X is defined as the derivative of its CDF. It is denoted by:

$$f_x(x) = \frac{d}{dx} F_x(x).$$

8. List the properties of PDF.

1. $f_x(x) \ge 0$
2. $\int_{-\infty}^{+\infty} f_x(x)\, dx = 1$
3. $\int_b^a f_x(x)\, dx = P(a < X \le b)$
4. In general, $P(X \in A) = \int_A f_x(x)\, dx$
5. $F_x(x) = \int_{-\infty}^{x} f_x(u)\, du.$

9. Define mean of a random variable.

The mean of the random variable X is defined as:

$$E[X] = \int_{-\infty}^{\infty} x\, f_x(x)\, dx \quad \text{(Continuous)}$$

$$\mu_x = E[X] = \sum_x x\, P[X = x] \quad \text{(Discrete)}$$

10. Define variance of a random variable.

The variance of the random variable X is defined as:

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 \, f_x(x)\, dx \quad \text{(Continuous)}$$

$$\sigma_X^2 = \sum_x (x-\mu_X)^2 \, P[X=x] \quad \text{(Discrete)}$$

11. Define covariance of a random variable.

The covariance of two random variables X and Y is defined as the expectation of the product of the two random variables given by:

$$\text{Cov}(X,Y) = E[XY] - \mu_X \, \mu_Y$$

12. Define correlation coefficient.

The correlation coefficient of two random variable is defined as:

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \, \sigma_Y}$$

where, the value of correlation coefficient ranges from -1 to 1.

13. When the two random variables are said to be uncorrelated?

The two random variables X and Y are said to be uncorrelated, if their covariance value is zero.

$$\text{Cov}(X,Y) = 0$$

14. Give the PMF of binomial random variable.

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad 0 \le i \le n$$

15. Give the mean and variance of binomial random variable.

Mean of binomial random variable is given by, $\mu = E(X) = np$

Variance of binomial random variable is given by, $\sigma^2 = np(1-p)$.

16. What is the importance of binomial random variable in the communication?

Binomial random variable can be used to model the total number of bits received in error, when sequence of n bits is transmitted over a channelwith a bit-error probability of p.

17. Give the PDF for uniform random variable.

This is a continuous random variable taking values between a and b with equal probabilities for intervals of equal length. The probability density function is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{Otherwise} \end{cases}$$

18. What is the importance of uniform random variable in the communication?

The phase of a received sinusoid carrier and quantization errors are usually modelled as a uniform random variable.

19. Give the PDF for Gaussian random variable.

The Gaussian or normal random variable is a continuous random variable described by the density function as :

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

20. What is the significance of Gaussian random variable?

- The Gaussian random variable is the most important and frequently encountered random variable in communication systems. The reason is that thermal noise, which is the major source of noise in communication, has a Gaussian distribution.
- In robotics, Gaussian PDF is used to statistically characterize sensor measurements, robot locations and map representations.

21. List the properties of Gaussian random variable.

- The sum of two independent Gaussian random variables is also a Gaussian randomvariable.
- The weighted sum of N independent Gaussian random variables is a Gaussian randomvariable.
- If two Gaussian random variables have zero covariance (uncorrelated), they are also independent.

22. State Central Limit Theorem. (May/June 2016)(Nov/Dec 2016)

Central limit theorem states that the normalized distribution of the sum of independent, identically distributed random variables approaches a Gaussian distribution as the number of random variables increases, regardless of the individual distributions.

23. List the applications of Central Limit Theorem.

- Signal processing
- Channel modelling
- Finance
- Population statistics
- Hypothesis testing
- Engineering research

24. Define Random Process.

A random process is defined as rule which assigns a function of time to each outcome‘s’ of a random experiment.

25. Differentiate random process from random variable.

Random variable is a mapping of event outcome to real numbers, whereas random process is the mapping of event outcome to signal waveforms. Random process is a function of time, but random variable is not a function of time.

26. What are the types of random process?

Based on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into discrete random sequence, continuous random sequence, discrete random process and continuous random process.

Based on the stationarity, a random process can be classified into stationary and non-stationary random process.

27. Define stationary random process.

A random process whose statistical characteristics do not change with time is classified as stationary random process or stationary process.

28. What is strict-sense stationary?

The random process X(t) is said to be Stationary in the Strict Sense (SSS) or strictly stationary if its statistics are invariant to a shift of origin:

$$F_{X(t_1+\tau),\ldots\ldots,X(t_k+\tau)}(x_1,\ldots\ldots.,x_k)=F_{X(t_1),\ldots\ldots X(t_k)}(x_1,\ldots\ldots.,x_k)$$

29. Define mean of the random process.

The mean of process X(t) is defined as the expectation of the random variable obtained by observing the process at some time t given by:

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x\, f_{x(t)}(x)\, dx \cdot$$

30. Define autocorrelation of the random process. (May/June 2016)

Autocorrelation of the process X(t) is given by the expectation of the product of two random variables X(t_1) and X(t_2) obtained by observing the process X(t) at times t_1 and t_2 respectively.

$$R_{XX}(t_1, t_2) = E[X(t_1)\,X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_{X(t_1),\, X(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

31. List the properties of autocorrelation function.

- The mean square value of a process is equal to the value of autocorrelation at $\tau = 0$.

$$R_{XX}(0) = E[X^2(T)]$$

- The autocorrelation function is an even function of τ.

$$R_{XX}(\tau) = R_{XX}(-\tau)$$

- The autocorrelation function is maximum at $\tau = 0$.

$$|R_{XX}(\tau)| \le R_{XX}(0)$$

32. Define autocovariance of the random process.

The autocovariance function of X(t) is defined as:

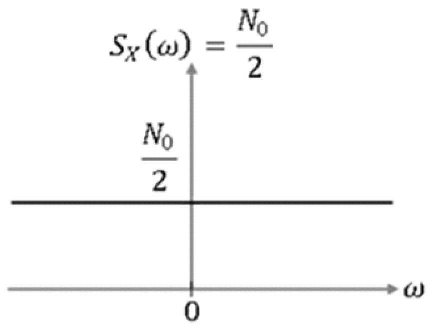
$$C_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1) - m_X^2$$

33. Define white noise process.

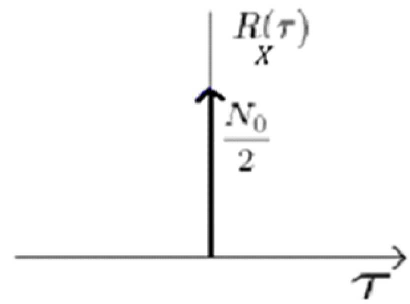
A random process is said to be white noise process X(t), if it is zero mean and power spectral density is defined as:

$$S_X(\omega) = N_0/2, \text{ for all frequencies}$$

34. Draw the PSD and autocorrelation function of white noise.



PSD of white noise



Autocorrelation of white noise

35. When a random process is said to be white Gaussian noise process?

The random process X(t) is said to be white Gaussian noise process, if X(t) is a stationary Gaussian random process with zero mean and flat power spectral density.

36. What is Wide-Sense Stationary? (Apr/May 2017)

A random process is called wide-sense stationary (WSS) if its

- Mean is constant.
- Autocorrelation depends only on the time difference.

37. Define Power Spectral Density of the random process.

The distribution of the power of the random processat different frequencies is the Power Spectral Density or Power Spectrum of the random process.

38. Give the power spectral density equation of a random process X.

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-j2\pi f \tau_1)\, d\tau_1$$

39. List the properties of power spectral density.

- The zero-frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function.
- The mean square value of a stationary process equals the total area under the graph of the power spectral density.

40. What is ergodicity?

A random process is said to be ergodic if time averages are the same for all sample functions and equal to the corresponding ensemble averages.

41. When a process is ergodic in mean?

A stationary process is called ergodic in the mean if:

$$\lim_{T\to\infty} m_x(T) = m_x$$

$$\lim_{T\to\infty} \text{var}[m_x(T)] = 0$$

42. When a process is ergodic in autocorrelation?

A stationary process is calledergodic in the autocorrelation function if,

$$\lim_{T\to\infty} R_{xx}(\tau, T) = R_{xx}(\tau)$$

$$\lim_{T\to\infty} \text{Var}[R_{xx}(\tau, T)] = 0$$

43. Write Einstein-Wiener-Khintchine relations. (Nov/Dec 2016) (Apr/May 2017)

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-j2\pi f\tau)\,d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\exp(j2\pi f\tau)\,df$$

44. Give the importance of Wiener-Khintchine relations.

For a stationary process, the power spectral density can be obtained from the Fourier transform of the autocorrelation function.

45. What are the advantages of Gaussian process?

- Gaussian process has many properties that make ***analytic results possible***.
- Random processes produced by physical phenomena are often such that a Gaussian model is appropriate. Further the use of Gaussian model is confirmed by experiments.

46. List the properties of Gaussian process.

- If a Gaussian process is stationary, then the process is also strictly stationary.
- If a Gaussian process X(t) is passed through LTI filter, then the random process Y(t) at the output of the filter is also Gaussian process.

PART – B

1. Let X and Y be real random variables with finite second moments. Prove the Cauchy-Schwartz inequality $(E[XY])^2 \leq E[X^2]E[Y^2]$. **(Apr/May 2015)**
2. Differentiate SSS with that of WSS process. **(Apr/May 2015)**
3. Let $X(t)$ and $Y(t)$ be both zero-mean and WSS random processes. Consider the random process $Z(t) = X(t) + Y(t)$. Determine the autocorrelation and power spectrum of $Z(t)$ if $X(t)$ and $Y(t)$ are jointly WSS. **(Apr/May 2015)**
4. Let $X(t) = A\cos(\omega t + \phi)$ and $Y(t) = A\sin(\omega t + \phi)$ where A and ω are constants and ϕ is a uniform random variable $[0, 2\pi]$. Find the cross correlation of $X(t)$ and $Y(t)$.

   (Apr/May 2015)(May/June 2016)
5. In a binary communication system, let the probability of sending a 0 and 1 be 0.3 and 0.7 respectively. Let us assume that a 0 being transmitted, the probability of it being received as 1 is 0.01 and the probability of error for a transmission of 1 is 0.1.

   (i) What is the probability that the output of this channel is 1?

   (ii) If a 1 is received then what is the probability that the input to the channel was 1?
   (Nov/Dec 2015)
6. What is CDF and PDF? State their properties. Also discuss them in detail by giving examples of CDF and PDF for different types of random variables. **(Nov/Dec 2015)**
7. Explain in detail about the transmission of a random process through a linear time invariant filter. **(May/June 2016)(Nov/Dec 2016)**
8. When a random process is said to be strict sense stationary (SSS), wide sense stationary (WSS) and ergodic process? **(May/June 2016)(Nov/Dec 2016)**
9. Given a random process, $X(t) = A\cos(\omega t + \mu)$ where A and ω are constants and μ is a uniformly distributed random variable. Show that $X(t)$ is ergodic in both mean and autocorrelation. **(May/June 2016)**
10. Define the following: Mean, Correlation, Covariance and Ergodicity. **(Nov/Dec 2016)**
11. What is a Gaussian random process and mention its properties. **(Nov/Dec 2016)**
12. Consider two linear filters connected in cascade as shown in fig.1. Let $X(t)$ be a stationary

process with auto correlation function $R_x(\tau)$, the random processappearing at the first input filter is $V(t)$ and the second filter output is $Y(t)$.

(a) Find the autocorrelation function of $Y(t)$.

(b) Find the cross correlation function $R_{vy}(\tau)$ of $V(t)$ and $Y(t)$. **(Apr/May 2017)**

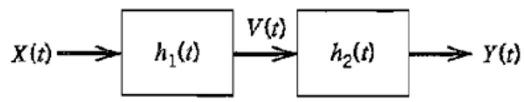


Fig. 1

13. The amplitude modulated signal is defined as $X_{AM}(t) = Am(t)\cos(\omega_c t + \theta)$ where $m(t)$ is the baseband signal and $A\cos(\omega_c t + \theta)$ is the carrier. The baseband signal $m(t)$ is modeled as a zero mean stationary random process with the autocorrelation function $R_{xx}(\tau)$ and the PSD $G_x(f)$. The carrier amplitude A and frequency ω_c areassumed to be constant and the initial carrier phase θ is assumed to be random uniformly distributed in the interval $(-\pi, \pi)$. Furthermore, $m(t)$ and θ are assumed to be independent.

(i) Show that $X_{AM}(t)$ is Wide Sense Stationary

(ii) Find PSD of $X_{AM}(t)$. **(Apr/May 2017)**

❒ ❒ ❒