JEPPIAAR INSTITUTE OF TECHNOLOGY
"Self-Belief| Self Discipline | Self Respect"

DEPARTMENT
OF
ELECTRICAL AND ELECTRONICS ENGINEERING

LECTURE NOTES

EE8451- ELECTROMAGENTIC FIELDS

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If one side of the interface, as shown in fig 5.4, is a perfect electric conductor, say region 2, a surface current $\vec{J_S}$ can exist even though $\vec{E}$ is zero as $\sigma = \infty$.
Thus eqn 5.27(a) and (c) reduces to

$$\widehat{a_n} \times \vec{H} = \vec{J_S} \qquad (5.28(a))$$
$$\widehat{a_n} \times \vec{E} = 0 \qquad (5.28(b))$$

Wave equation and their solution:

From equation 5.25 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{D} = \vec{\rho}$$
$$\nabla \cdot \vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (5.29(a))$$
$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (5.29(b))$$
$$\nabla \cdot \vec{E} = 0 \qquad (5.29(c))$$
$$\nabla \cdot \vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot \left(\nabla \cdot \vec{A}\right) - \nabla^2 A$$

We can write from 5.29(b)

$$\nabla \times \nabla \times \vec{E} = \nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$
$$= -\nabla \times \left(\mu \frac{\partial \vec{H}}{\partial t} \right)$$

or $$\nabla \cdot \left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right)$$

Substituting $\nabla\times\vec{H}$ from 5.29(a)

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free medium $\nabla\cdot\vec{E}=0$ (eqn 5.29(c))

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad (5.30)$$

In the same manner for equation eqn 5.29(a)

$$\begin{aligned}\nabla\times\nabla\times\vec{H}&=\nabla\cdot\left(\nabla\cdot\vec{H}\right)-\nabla^2\vec{H}\\&=\sigma\left(\nabla\times\vec{E}\right)+\varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)\\&=\sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)\end{aligned}$$

Since $\nabla\cdot\vec{H}=0$ from eqn 5.29(d), we can write

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right) \qquad (5.31)$$

These two equations

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. For example, if we consider a Cartesian co ordinate system, $\vec{E}$ and $\vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space, i.e. $\sigma = 0$, $\mu = \mu_0$, and $\varepsilon = \varepsilon_0$. The wave eqn in equations 5.30 and 5.31 reduces to

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \qquad (5.32(a))$$

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \qquad (5.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system a special case where $\vec{E}$ and $\vec{H}$ are considered to be independent in two dimensions, say $\vec{E}$ and $\vec{H}$ are assumed to be independent of *y* and *z*. Such waves are called plane waves.

From eqn (5.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalar equations

$$\frac{\partial^2 \vec{E_x}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_x}}{\partial t^2} \right) \qquad (5.33(a))$$

$$\frac{\partial^2 \vec{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_y}}{\partial t^2} \right) \qquad (5.33(b))$$

$$\frac{\partial^2 \vec{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_z}}{\partial t^2} \right) \qquad (5.33(c))$$

Since we have $\nabla \cdot \vec{E} = 0$,

$$\therefore \frac{\partial \overrightarrow{E_x}}{\partial x} + \frac{\partial \overrightarrow{E_y}}{\partial y} + \frac{\partial \overrightarrow{E_z}}{\partial z} = 0 \qquad (5.34)$$

As we have assumed that the field components are independent of y and z eqn (5.34)
reduces to

$$\frac{\partial E_x}{\partial x} = 0 \qquad (5.35)$$

i.e. there is no variation of E_X in the x direction.

Further, from 5.33(a), we find that $\frac{\partial E_x}{\partial x} = 0$ implies $\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires any three of the conditions to be satisfied: (i) $E_X=0$, (ii)E_X = constant, (iii)E_X increasing uniformly with time.

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_X is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (*E* or *H*) acting along x.

Without loss of generality let us now consider a plane wave having E_Y component only (Identical results can be obtained for E_Z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (5.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (5.37)$$

where $v_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (5.37) satisfies wave eqn (5.36) can be verified by substitution.
$f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$ corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (5.37) is zero and
the solution can be written as

$$E_y = f_1\left(x - v_0 t\right) \qquad (5.38)$$

Such a wave motion is graphically shown in fig 5.5 at two instances of time t_1 and t_2.

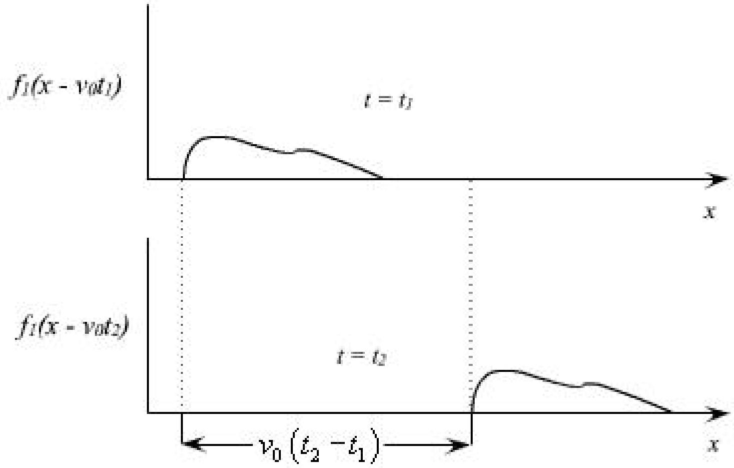


Fig 5.5 : Traveling wave in the + x direction

Let us now consider the relationship between E and H components for the forward traveling wave.

Since $\vec{E} = \widehat{a_y} E_y = \widehat{a_y} f_1\left(x - v_0 t\right)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \widehat{a_z} \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (5.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (5.39)$$

and from (5.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\widehat{a_y} \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t} \qquad (5.40)$$

Substituting E_y from (5.38)

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'\left(x - v_0 t\right)$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'\left(x - v_0 t\right)$$

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'\left(x - v_0 t\right)dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

The constant of integration means that a field independent of x may also exist. However, this field will not be a part of the wave motion.

Hence $$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y \qquad (5.41)$$

which relates the *E* and *H* components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$ is called the characteristic or intrinsic impedance of the free space

Harmonic fields

In the previous section we introduced the equations pertaining to wave propagation and discussed how the wave equations are modified for time harmonic case. In this section we discuss in detail a particular form of electromagnetic wave propagation called 'plane waves'. **The Helmhotz Equation:**

In source free linear isotropic medium, Maxwell equations in phasor form are,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \nabla \times \vec{H} = 0$$

$$\therefore \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \times \vec{E}\right) - \nabla^2 \vec{E} = -j\omega\mu\nabla \times \vec{H}$$

or, $$-\nabla^2 \vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E})$$

$$\text{or, } \nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = 0$$

$$\text{or, } \nabla^2 \vec{E} + k^2 \vec{E} = 0 \text{ where } k = \omega\sqrt{k\varepsilon}$$

An identical equation can be derived for $\vec{H}$.

$$\text{i.e., } \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

These equations

$$\left.\begin{aligned} &\nabla^2 \vec{E} + k^2 \vec{E} = 0 \ldots\ldots\ldots\ldots\ldots(a) \\ \& \quad &\nabla^2 \vec{H} + k^2 \vec{H} = 0 \ldots\ldots\ldots\ldots\ldots(b) \end{aligned}\right\} \ldots\ldots(6.1)$$

are called homogeneous vector Helmholtz's equation.

$k = \omega\sqrt{\mu\varepsilon}$ is called the wave number or propagation constant of the medium.

Plane waves in Lossless medium:

In a lossless medium, ε and μ are real numbers, so k is real.

In Cartesian coordinates each of the equations 6.1(a) and 6.1(b) are equivalent to three scalar Helmholtz's equations, one each in the components E_X, E_y and E_Z or H_X, H_y, H_Z.

For example if we consider E_X component we can write

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.2)$$

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wavefront or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical JITnarios.

Let us consider a plane wave which has only E_X component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation (6.2) reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \qquad \text{..........(6.3)}$$

The solution to this equation can be written as

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jkz} + E_0^- e^{jkz} \qquad \text{..........(6.4)}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions). In the time domain, $\varepsilon_X(z,t) = \mathrm{Re}(E_x(z)e^{j\omega t})$

$$\varepsilon_X(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz) \qquad \text{..........(6.5)}$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_X^+(z,t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^+(z,t)$ for several values of t is shown in the Figure 6.1.

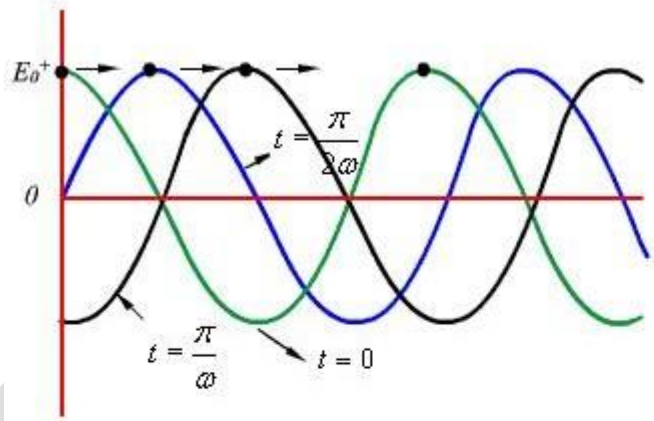


Figure 6.1: Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$$\omega(t+\Delta t) - k(z+\Delta z) = \text{constant} = \omega t - \beta z$$

Or, $\omega \Delta t = k \Delta z$

Or, $\dfrac{\Delta z}{\Delta t} = \dfrac{\omega}{k}$

When $\Delta t \rightarrow 0$,

we write $\lim_{\Delta t \rightarrow 0} \dfrac{\Delta z}{\Delta t} = \dfrac{dz}{dt} =$ phase velocity v_p .

$$\therefore v_p = \frac{\omega}{k} \quad \text{....................................(6.6)}$$

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

Then $v_p = \dfrac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $(\omega t - kz) - \left[\omega t - k(z+\lambda)\right] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \dfrac{2\pi}{k}$

Substituting $k = \dfrac{\omega}{v_p}$,

$$\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$$

or, $\lambda f = v_p$ ....(6.7)

Thus wavelength λ also represents the distance covered in one oscillation of the wave. Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y \quad ....(6.8)$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium. When the wave travels in free space

$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$ is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^{+}(z,t) = \hat{a}_y \frac{E_0^{+}}{\eta}\cos(\omega t - \beta z) \quad \text{........... (6.9)}$$

Which represents the magnetic field of the wave traveling in the +z

direction. For the negative traveling wave,

$$\vec{H}^{-}(z,t) = -a_y \frac{E_0^{+}}{\eta}\cos(\omega t + \beta z) \quad \text{...........(6.10)}$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig 6.2.

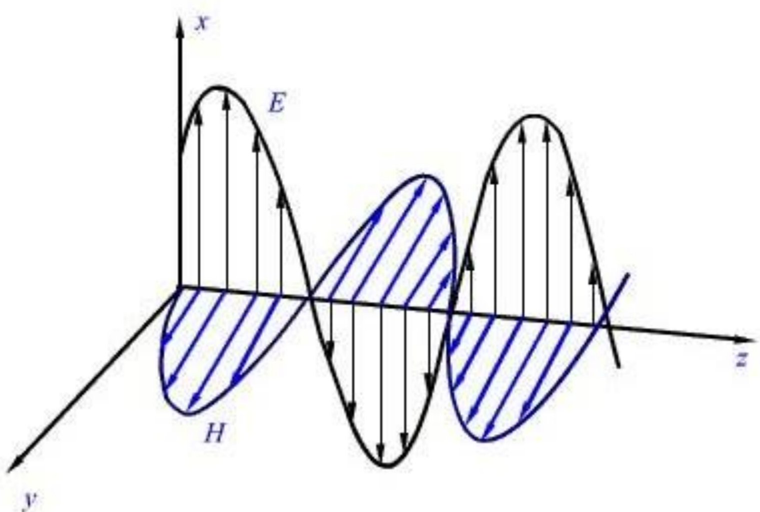


Figure 6.2 : E & H fields of a particular plane wave at time t.

TEM Waves:

So far we have considered a plane electromagnetic wave propagating in the z-direction. Let us now consider the propagation of a uniform plane wave in any arbitrary direction that doesn't necessarily coincides with an axis.

For a uniform plane wave propagating in z-direction

$\vec{E}(z) = E_0 e^{-jkz}$, E_0 is a constant vector..........................

(6.11) The more general form of the above equation is

$\vec{E}(x,y,z) = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z}$ ....

(6.12) This equation satisfies Helmholtz's equation $\nabla^2 \vec{E} + k^2 \vec{E} = 0$ provided,

$$k_x^{\,2} + k_y^{\,2} + k_z^{\,2} = k^2 = \omega^2 \mu \varepsilon \quad \text{.......................... (6.13)}$$

We define wave number vector $\vec{k} = \hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z = k\hat{a}_n$ ....

(6.14) And radius vector from the origin

$$\vec{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z \quad \text{.......................... (6.15)}$$

Therefore we can write

$$\vec{E}(\vec{r}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{r}} \quad \text{.......................... (6.16)}$$

Here $\hat{a}_n \cdot \vec{r}$ =constant is a plane of constant phase and uniform amplitude just in the case of $\vec{E}(z) = \vec{E}_0 e^{-jkz}$,

z =constant denotes a plane of constant phase and uniform

amplitude. If the region under consideration is charge free,

$$\nabla \cdot \vec{E} = 0$$

$$\therefore \nabla \cdot \left(\vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} \right) = 0$$

Using the vector identity $\nabla \cdot \left(f\vec{A} \right) = \vec{A}\nabla f + f\nabla \cdot \vec{A}$ and noting that $\vec{E}_0$ is constant we can write,

$$\vec{E}_0 . \nabla \left(e^{-jk\hat{a}_n . \vec{r}} \right) = 0$$

$$or,\ \vec{E}_0 . \left[\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \right) e^{-j(k_x x + k_y y + k_z z)} \right] = 0$$

$$or,\ \vec{E}_0 . \left(-jk\,\hat{a}_n e^{-jk\hat{a}_n . \vec{r}} \right) = 0$$

$$\vec{E}_0 . \hat{a}_n = 0 \quad ....(6.17)$$

i.e., $\vec{E}_0$ is transverse to the direction of the propagation.

The corresponding magnetic field can be computed as follows:

$$\vec{H}\left(\vec{r}\right) = -\frac{1}{j\omega\mu} \nabla \times \vec{E}\left(\vec{r}\right) = -\frac{1}{j\omega\mu} \nabla \times \left(\vec{E}_0 e^{-j\vec{k}.\vec{r}} \right)$$

Using the vector identity,

$$\nabla \times \left(\psi \vec{A} \right) = \psi \nabla \times \vec{A} + \nabla \psi \times \vec{A}$$

Since $\vec{E}_0$ is constant we can write,

$$\vec{H}(\vec{r}) = -\frac{1}{j\omega\mu} \nabla e^{-j\vec{k}.\vec{r}} \times \vec{E}_0$$

$$= -\frac{1}{j\omega\mu} \left[-jk\,\hat{a}_n \times \vec{E}_0 e^{-jk\hat{a}_n . \vec{r}} \right]$$

$$= \frac{k}{\omega\mu} \hat{a}_n \times \vec{E}\left(\vec{r}\right)$$

$$\vec{H}(\vec{r}) = \frac{1}{\eta} \hat{a}_n \times \vec{E}\left(\vec{r}\right) \quad ....(6.18)$$

Where η is the intrinsic impedance of the medium. We observe that $\vec{H}(\vec{r})$ is perpendicular to both $\hat{a}_n$ and $\vec{E}\left(\vec{r}\right)$. Thus the electromagnetic wave represented by $\vec{E}\left(\vec{r}\right)$ and $\vec{H}(\vec{r})$ is a TEM wave.

Plane waves in a lossy medium**:**

In a lossy medium, the EM wave looses power as it propagates. Such a medium is conducting with conductivity σ and we can write:

$$\begin{aligned}\nabla\times\vec{H} &= \vec{J} + j\omega\varepsilon\vec{E} = (\sigma + j\omega\varepsilon)\vec{E} \\ &= j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\vec{E} \\ &= j\omega\varepsilon_c\vec{E}\end{aligned} \qquad \text{.....................(6.19)}$$

Where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$ is called the complex permittivity.

We have already discussed how an external electric field can polarize a dielectric and give rise to bound charges. When the external electric field is time varying, the polarization vector will vary with the same frequency as that of the applied field. As the frequency of the applied filed increases, the inertia of the charge particles tend to prevent the particle displacement keeping pace with the applied field changes. This results in frictional damping mechanism causing power loss.

In addition, if the material has an appreciable amount of free charges, there will be ohmic losses. It is customary to include the effect of damping and ohmic losses in the imaginary part of ε_c. An equivalent conductivity $\sigma = \omega\varepsilon''$ represents all losses.

The ratio $\dfrac{\varepsilon''}{\varepsilon'}$ is called loss tangent as this quantity is a measure of the power loss.

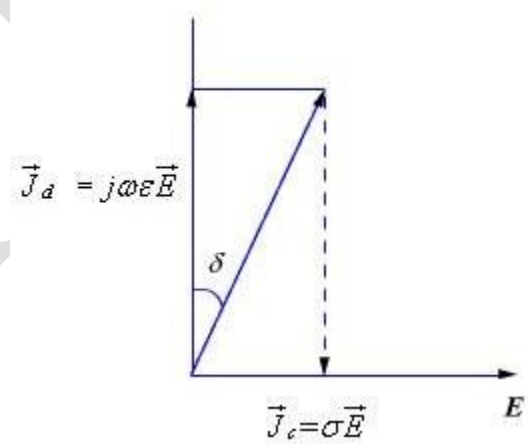


Fig 6.3 : Calculation of Loss Tangent

With reference to the Fig 6.3,

$$\tan\delta = \frac{|\vec{J}_c|}{|\vec{J}_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'} \quad \text{.......................... (6.20)}$$

where $\vec{J}_c$ is the conduction current density and $\vec{J}_d$ is displacement current density. The loss tangent gives a measure of how much lossy is the medium under consideration. For a good dielectric medium $(\sigma << \omega\varepsilon)$, $\tan\delta$ is very small and the medium is a good conductor if $(\sigma \gg \omega\varepsilon)$. A material may be a good conductor at low frequencies but behave as lossy dielectric at higher frequencies.

For a source free lossy medium we can write

$$\left.\begin{aligned} \nabla\times\vec{H} &= (\sigma + j\omega\varepsilon)\vec{E} & \nabla.\vec{H} &= 0 \\ \nabla\times\vec{E} &= -j\omega\mu\vec{H} & \nabla.\vec{E} &= 0 \end{aligned}\right\} \quad \text{...........................(6.21)}$$

$$\nabla\times\nabla\times\vec{E} = \nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} = -j\omega\mu\nabla\times\vec{H} = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$or, \nabla^2\vec{E} - \gamma^2\vec{E} = 0 \quad \text{.................... (6.22)}$$

Where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

Proceeding in the same manner we can write,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

$$\gamma = \alpha + i\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

is called the propagation constant.

The real and imaginary parts α and β of the propagation constant η can be computed as follows:

$$\gamma^2 = (\alpha + i\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$or, \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

And $$\alpha\beta = \frac{\omega\mu\sigma}{2}$$

$$\therefore \alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$or, 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon + \omega^4\mu^2\varepsilon^2 = \omega^2\mu^2\sigma^2 + \omega^4\mu^2\varepsilon^2$$

$$or, \left(2\alpha^2 + \omega^2\mu\varepsilon\right)^2 = \omega^4\mu^2\varepsilon^2\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

$$or, \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$ .... (6.23a)

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

Similarly.. (6.23b)

Let us now consider a plane wave that has only x -component of electric field and propagate along z .

$$\therefore \vec{E}_x(z) = \left(E_0^+ e^{-\gamma z} + E_0^- e^{-\gamma z}\right)\hat{a}_x$$ .... (6.24)

Considering only the forward traveling wave

$$\vec{\varepsilon}(z,t) = \mathrm{Re}\left(E_0^+ e^{-\gamma z} e^{j\omega t}\right)\hat{a}_x$$

$$= E_0^+ e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_x$$ .... (6.25)

Similarly, from $\vec{\mathcal{H}} = -\frac{1}{j\omega\mu}\nabla \times \vec{E}$, we can find

$$\vec{\mathcal{H}}(z,t) = \frac{E_0}{\eta}e^{-\alpha z}\cos\left(\omega t - \beta z\right)\hat{a}_y$$ .... (6.26)

Where $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| e^{j\theta_\eta}$

$$\therefore \vec{\mathcal{H}} = \frac{E_0}{|\eta|}e^{-\alpha z}\cos\left(\omega t - \beta z - \theta_\eta\right)\hat{a}_y$$ .... (6.27)

From (6.25) and (6.26) we find that as the wave propagates along z, it decreases in amplitude by a factor $e^{-\alpha z}$. Therefore α is known as attenuation constant. Further $\vec{E}$ and $\vec{H}$

are out of phase by an angle θ_η.

For low loss dielectric, $\dfrac{\sigma}{\omega\varepsilon} << 1$, i.e., $\varepsilon'' << \varepsilon'$.

Using the above condition approximate expression for α and β can be obtained as follows:

$$\gamma = \alpha + i\beta = j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{\varepsilon'}\right]^{1/2}$$

$$\cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{1}{2}\frac{\varepsilon''}{\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\left.\begin{aligned}\alpha &= \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}\\ \beta &= \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]\end{aligned}\right\} \text{........(6.28)}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) \text{ .... (6.29)}$$

& phase velocity

$$v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \text{ .... (6.30)}$$

For good conductors $\dfrac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \cong j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} \text{ .... (6.31)}$$

We have used the relation

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

From (6.31) we can write

$$\alpha + i\beta = \sqrt{\pi f \mu \sigma} + j\sqrt{\pi f \mu \sigma}$$

$$\therefore \alpha = \beta = \sqrt{\pi f \mu \sigma} \quad \text{..............} \ (6.32)$$

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}}$$

$$\cong \sqrt{\frac{\mu}{\varepsilon}\frac{j\omega\varepsilon}{\sigma}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\frac{\alpha}{\sigma} \quad \text{....................} \ (6.33)$$

And phase velocity

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad \text{....................} \ (6.34)$$

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla.\left(\vec{E}\times\vec{H}\right)=\vec{H}.\nabla\times\vec{E}-\vec{E}.\nabla\times\vec{H}$$

the above curl equations we can write

$$\nabla.\left(\vec{E}\times\vec{H}\right)=-\vec{H}.\frac{\partial\vec{B}}{\partial t}-\vec{E}.\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right)=-\vec{H}.\frac{\partial\vec{B}}{\partial t}-\vec{E}.\vec{J}-\vec{E}.\frac{\partial\vec{D}}{\partial t}$$ ....(6.35)

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right)$$ and $$\vec{E}.\vec{J}=\sigma E^2$$

$$\therefore\nabla.\left(\vec{E}\times\vec{H}\right)=-\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2+\frac{1}{2}\mu H^2\right)-\sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S\left(\vec{E}\times\vec{H}\right).d\vec{S}=-\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2+\frac{1}{2}\mu H^2\right)dV-\int_V\sigma E^2dV$$ ....(6.36)

The term $\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2+\frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S\left(\vec{E}\times\vec{H}\right).d\vec{S}=\oint_S \vec{P}.d\vec{S}$ where $\vec{P}=\vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z)=\hat{a}_x E_x(z)=\hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t)=\text{Re}\left[\vec{E}(z)e^{j\omega t}\right]=E_0\cos(\omega t-\beta z)\hat{a}_x \quad\text{................................(6.37)}$$

when *E0* is real.

Let us consider two instanteneous quantities A and B such that

$$A=\text{Re}\left(Ae^{j\omega t}\right)=|A|\cos(\omega t+\alpha)$$

$$B=\text{Re}\left(Be^{j\omega t}\right)=|B|\cos(\omega t+\beta)$$

where A and B are the phasor quantities.

i.e,

$$A=|A|e^{j\alpha}$$

$$B=|B|e^{j\beta}$$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right] \quad \text{..............................(6.39)}$$

Since *A* and *B* are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta) \quad \text{....................................(6.40)}$$

Further, considering the phasor quantities *A* and *B*, we find that

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}(AB^*) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}(AB^*) \quad \text{...(6.41)}$$

The poynting vector $\vec{P} = \vec{E} \times \vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x(E_y H_z - E_z H_y) + \hat{a}_y(E_z H_x - E_x H_z) + \hat{a}_z(E_x H_y - E_y H_x) \quad \text{...................................(6.42)}$$

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (6.42) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6.41)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z) \times H_y(z)\right) \quad \text{.......................................(6.43)}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration. For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right) \qquad \text{.....................(6.44)}$$

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}\left(\vec{S}\right)$.

Electromagnetic Spectrum:

The polarisation of a plane wave can be defined as the orientation of the electric field vector as a function of time at a fixed point in space. For an electromagnetic wave, the specification of the orientation of the electric field is sufficent as the magnetic field components are related to electric field vector by the Maxwell's equations.

Let us consider a plane wave travelling in the +z direction. The wave has both E_X and E_Y components.

$$\vec{E} = \left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right)e^{-j\beta z} \qquad \text{..(6.45)}$$

The corresponding magnetic fields are given by,

$$\vec{H} = \frac{1}{\eta}\hat{a}_z\times\vec{E}$$

$$= \frac{1}{\eta}\hat{a}_z\times\left(\hat{a}_x E_{ox} + \hat{a}_y E_{oy}\right)e^{-j\beta z}$$

$$= \frac{1}{\eta}\left(-E_{oy}\hat{a}_x + E_{ox}\hat{a}_x\right)e^{-j\beta z}$$

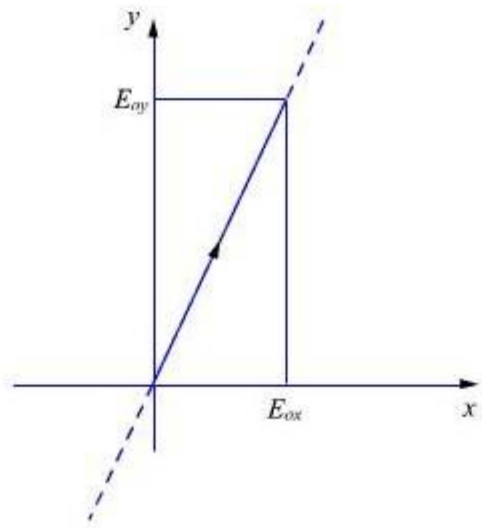
Depending upon the values of E_{OX} and E_{Oy} we can have several possibilities:

1. If $E_{Oy} = 0$, then the wave is linearly polarised in the x-direction.

2. If $E_{Oy} = 0$, then the wave is linearly polarised in the y-direction.

3. If E_{OX} and E_{Oy} are both real (or complex with equal phase), once again we get a

linearly polarised wave with the axis of polarisation inclined at an angle $\tan^{-1}\frac{E_{oy}}{E_{ox}}$, with respect to the x-axis. This is shown in fig 6.4.



emerging

Fig 6.4:Linear Polarisation

4. If *Eox* and *Eoy* are complex with different phase angles, $\vec{E}$ will not point to a single spatial direction. This is explained as follows:

Let $E_{ox} = |E_{ox}| e^{ja}$

$$E_{oy} = |E_{oy}| e^{jb}$$

Then,

$$E_x(z,t) = \mathrm{Re}\left[|E_{ox}| e^{ja} e^{-j\beta z} e^{j\omega t}\right] = |E_{ox}| \cos(\omega t - \beta z + a)$$

and $E_y(z,t) = \mathrm{Re}\left[|E_{oy}| e^{jb} e^{-j\beta z} e^{j\omega t}\right] = |E_{oy}| \cos(\omega t - \beta z + b)$ ....(6.46)

To keep the things simple, let us consider a =0 and $b = \frac{\pi}{2}$. Further, let us study the nature of the electric field on the z =0 plain.

From equation (6.46) we find that,

$$E_x(o,t) = |E_{ox}| \cos \omega t$$

$$E_y(o,t) = |E_{oy}|\cos\left(\omega t + \frac{\pi}{2}\right) = |E_{oy}|(-\sin\omega t)$$

$$\therefore \left(\frac{E_x(o,t)}{|E_{ox}|}\right)^2 + \left(\frac{E_y(o,t)}{|E_{oy}|}\right)^2 = \cos^2\omega t + \sin^2\omega t = 1 \quad \text{.....................................(6.47)}$$

and the electric field vector at z = 0 can be written as

$$\vec{E}(o,t) = |E_{ox}|\cos(\omega t)\hat{a}_x - |E_{oy}|\sin(\omega t)\hat{a}_y \quad \text{..(6.48)}$$

Assuming $|E_{ox}| > |E_{oy}|$, the plot of $\vec{E}(o,t)$ for various values of t is hown in figure 6.5.

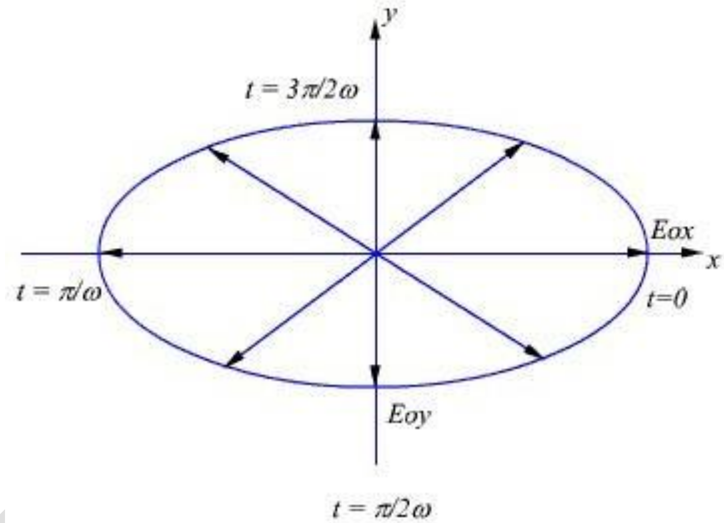


Figure 6.5:Plot of *E*(*o*,*t*)

From equation (6.47) and figure (6.5) we observe that the tip of the arrow representing electric field vector traces qn ellipse and the field is said to be elliptically polarised.

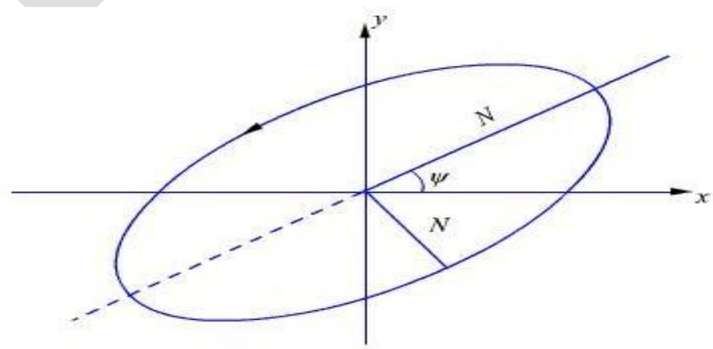


Figure 6.6: Polarisation ellipse

The polarisation ellipse shown in figure 6.6 is defined by its axial ratio(M/N, the ratio of semimajor to semiminor axis), tilt angle ψ (orientation with respect to xaxis) and sense of rotation(i.e., CW or CCW).

Linear polarisation can be treated as a special case of elliptical polarisation, for which the axial ratio is infinite.

In our example, if $|E_{ox}| = |E_{oy}|$, from equation (6.47), the tip of the arrow representing electric field vector traces out a circle. Such a case is referred to as Circular Polarisation. For circular polarisation the axial ratio is unity.

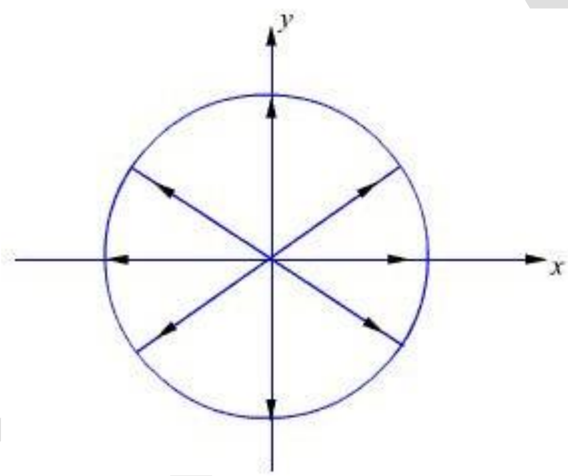


Figure 6.7:Circular Polarisation (RHCP)

Further, the circular polarisation is aside to be right handed circular polarisation (RHCP) if the electric field vector rotates in the direction of the fingers of the right hand when the thumb points in the direction of propagation-(same and CCW). If the electric field vector rotates in the opposite direction, the polarisation is asid to be left hand circular polarisation (LHCP) (same as CW).

In AM radio broadcast, the radiated electromagnetic wave is linearly polarised with the $\vec{E}$ field vertical to the ground(vertical polarisation) where as TV signals are horizontally polarised waves. FM broadcast is usually carried out using circularly polarised waves.

In radio communication, different information signals can be transmitted at the same frequency at orthogonal polarisation (one signal as vertically polarised other horizontally polarised or one as RHCP while the other as LHCP) to increase capacity. Otherwise, same signal can be transmitted at orthogonal polarisation to obtain diversity gain to improve reliability of transmission.

Behaviour of Plane waves at the inteface of two media:

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

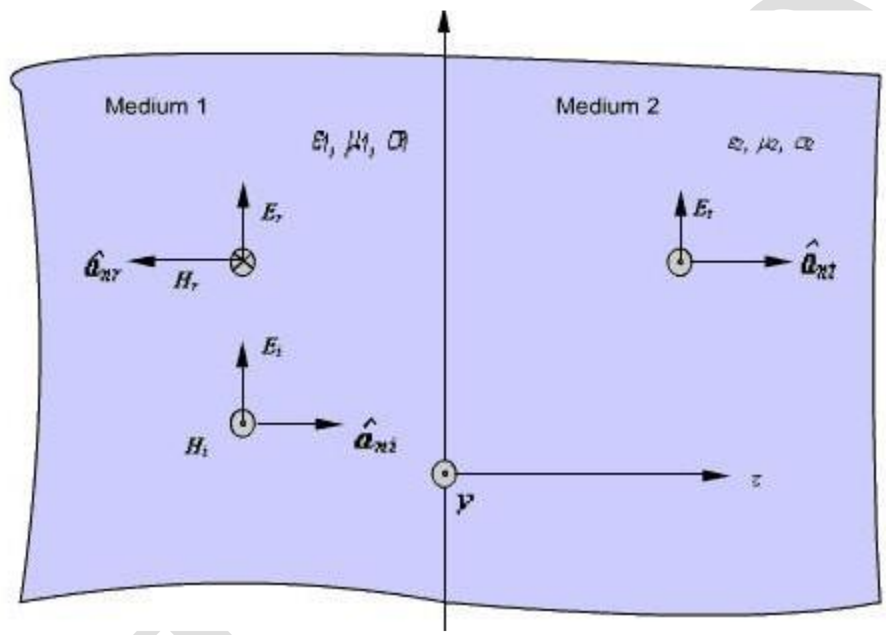


Fig 6.8 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.

Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively.

The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium

1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad \text{..................(6.49.a)}$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad \text{......................(6.49.b)}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\dfrac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at z =0, the incident wave will undergo partial reflection and partial transmission.
The reflected wave will travel along $-\hat{a}_z$ in medium

1. The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z}\,\hat{a}_x \quad \text{..(6.50a)}$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z}\,\hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z}\,\hat{a}_y \quad \text{.........(6.50b)}$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z}\,\hat{a}_x \quad \text{...(6.51a)}$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z}\,\hat{a}_y \quad \text{...(6.51b)}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1, $\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2, $\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$$

& $$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$$

From equation 6.49 to 6.51 we get,

$$E_{io} + E_{ro} = E_{to} \qquad \text{...(6.52a)}$$

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{E_{to}}{\eta_2} \qquad \text{...(6.52b)}$$

Eliminating
Eto ,

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}\left(E_{io} + E_{ro}\right)$$

or, $$E_{io}\left(\frac{1}{\eta_1} - \frac{1}{\eta_2}\right) = E_{ro}\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$$

or, $$E_{ro} = \tau E_{io}$$

..............(6.53)

$$\tau = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

is called the reflection coefficient.

From equation (6.52), we can

write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

or, $$E_{to} = \frac{2\eta_2}{\eta_1 + \eta_2} E_{io} = T E_{io}$$

$$T = \frac{2\eta_2}{\eta_1 + \eta_2}$$ ....(6.54)

is called the transmission coefficient. We observe that,

$$T = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$$ ....(6.55)

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \le |\tau| \le 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\eta_2 = 0$$

$$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$$
$$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$$

From (6.53) and (6.54)

$$\tau = -1$$

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\hat{a}_x - E_{io}e^{j\beta_1 z}\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$$

&

$$\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x \quad \text{.................................(6.56)}$$

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t \quad \text{...(6.57)}$$

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given *' t'*, both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

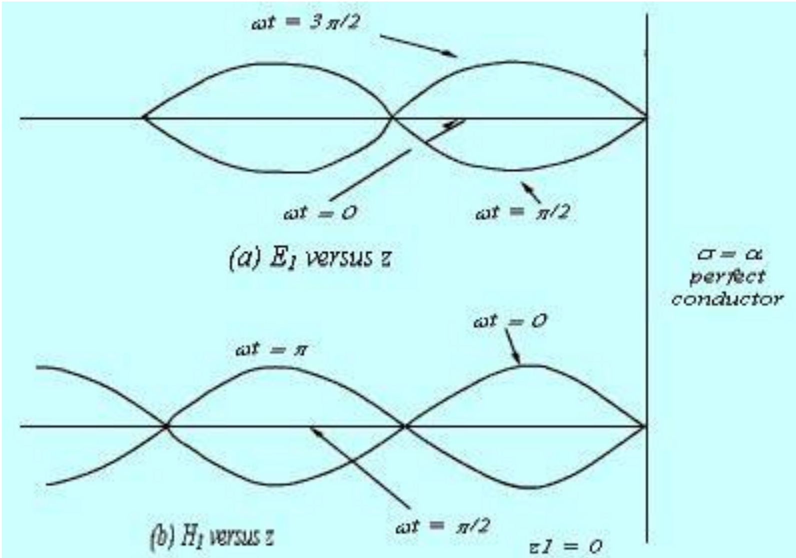


Figure 6.9: Generation of standing wave

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\right\} \text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\} \text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4}, \quad n = 0,1,2,....$$

$$\left.\right\} \quad ....(6.58)$$

Case2: Normal incidence on a plane dielectric boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.

From equation (6.49(a)) and equation (6.53) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad \text{..................(6.59)}$$

Let us consider the JITnario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0, \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad \text{..................(6.60)}$$

In this case both η_1 and η_1 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\\ &= \hat{a}_x E_{io}\left(\left(1+T\right)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right)\\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma\left(2j\sin\beta_1 z\right)\right)\end{aligned} \quad \text{..................(6.61)}$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$.

The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows.

The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \quad \text{..................(6.62)}$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}\left(1+T\right) \quad \text{..................(6.63)}$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $$z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$$, n = 0, 1, 2, 3 .... (6.64)

The minimum value of $\left|\vec{E}_1\right|$ is

$$\left|\vec{E}_1\right|_{min} = E_{io}\left(1-\Gamma\right)$$ ....(6.65)

And this occurs when

$$2\beta_1 z_{min} = -\left(2n+1\right)\pi$$

or $$z_{min} = -\left(2n+1\right)\frac{\lambda_1}{4}$$, n = 0, 1, 2, 3..........................(6.66) For $\eta_2 < \eta_1$ i.e. Γ *<0*

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1-\Gamma\right)$ which occurs at the z_{min} locations and the minimum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1+\Gamma\right)$ which occurs at z_{max} locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\frac{|E|_{max}}{|E|_{min}}$ can be written as

$$S = \frac{|E|_{max}}{|E|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$ ....(6.67)

The quantity *S* is called as the standing wave ratio.

As $0 \le |\Gamma| \le 1$ the range of *S* is given by $1 \le S \le \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$$ ....(6.68)

From (6.68) we find that $|\vec{H}_1|$ will be maximum at locations where $|\vec{E}_1|$ is minimum and vice versa.
In medium 2, the transmitted wave propagates in the + z direction.

Oblique Incidence of EM wave at an interface

So far we have discuss the case of normal incidence where electromagnetic wave traveling in a lossless medium impinges normally at the interface of a second medium. In this section we shall consider the case of oblique incidence. As before, we consider two cases

i. When the second medium is a perfect conductor.
ii. When the second medium is a perfect dielectric.

A plane incidence is defined as the plane containing the vector indicating the direction of propagation of the incident wave and normal to the interface. We study two specific cases when the incident electric field $\vec{E}_i$ is perpendicular to the plane of incidence (perpendicular polarization) and $\vec{E}_i$ is parallel to the plane of incidence (parallel polarization). For a general case, the incident wave may have arbitrary polarization but the same can be expressed as a linear combination of these two individual cases.

Oblique Incidence at a plane conducting boundary i. Perpendicular Polarization
The situation is depicted in figure 6.10.

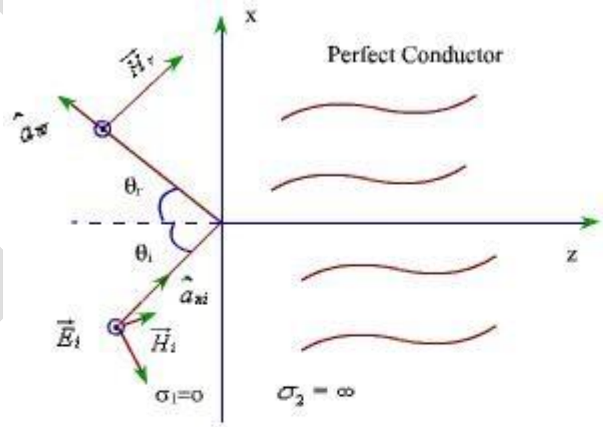


Figure 6.10: Perpendicular Polarization

As the EM field inside the perfect conductor is zero, the interface reflects the incident plane wave. $\hat{a}_{ni}$ and $\hat{a}_{nr}$ respectively represent the unit vector in the direction of propagation of the incident and reflected waves, θ_i is the angle of incidence and θ_r is the angle of reflection.

We find that

$$\hat{a}_{ni} = \hat{a}_z \cos\theta_i + \hat{a}_x \sin\theta_i$$
$$\hat{a}_{nr} = -\hat{a}_z \cos\theta_r + \hat{a}_x \sin\theta_r \quad \text{..........................(6.69)}$$

Since the incident wave is considered to be perpendicular to the plane of incidence, which for the present case happens to be xz plane, the electric field has only y-component.

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1 \bar{a}_{ni}\cdot\bar{r}}$$
$$= \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The corresponding magnetic field is given by

$$\vec{H}_i(x,z) = \frac{1}{\eta_1}\left[\hat{a}_n \times \vec{E}_i(x,z)\right]$$
$$= \frac{1}{\eta_1}\left[-\cos\theta_i \hat{a}_x + \sin\theta_i \hat{a}_z\right] E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{..........................(6.70)}$$

Similarly, we can write the reflected waves as

$$\vec{E}_r(x,z) = \hat{a}_y E_{ro} e^{-j\beta_1 \bar{a}_{nr} \cdot \bar{r}}$$
$$= \hat{a}_y E_{ro} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)} \qquad \text{...(6.71)}$$

Since at the interface z=o, the tangential electric field is zero.

$$E_{io} e^{-j\beta_1 x \sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = 0 \qquad \text{..(6.72)}$$

Consider in equation (6.72) is satisfied if we have

$$E_{ro} = -E_{io}$$
$$\text{and } \theta_i = \theta_r \qquad \text{.................................(6.73)}$$

The condition $\theta_i = \theta_r$ is Snell's law of reflection.

$$\therefore \vec{E}_r(x,z) = -\hat{a}_y E_{io} e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)} \qquad \text{..................................(6.74)}$$

$$\text{and } \vec{H}_r(x,z) = \frac{1}{\eta_1}\left[\hat{a}_{nr} \times \vec{E}_r(x,z)\right]$$
$$= \frac{E_{io}}{\eta_1}\left[-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i\right] e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)} \qquad \text{..................................(6.75)}$$

The total electric field is given by

$$\vec{E}_1(x,z) = \vec{E}_i(x,z) + \vec{E}_r(x,z)$$
$$= -\hat{a}_y 2jE_{io} \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} \qquad \text{..................................(6.76)}$$

Similarly, total magnetic field is given by

$$\vec{H}_1(x,z) = -2\frac{E_{io}}{\eta_1}\left[\hat{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i} + \hat{a}_z j \sin\theta_i \sin(\beta_1 z\cos\theta_i) e^{-j\beta_1 x\sin\theta_i}\right] \qquad \text{............................(6.77)}$$

From eqns (6.76) and (6.77) we observe that

1. Along z direction i.e. normal to the boundary
   y component of $\vec{E}$ and x component of $\vec{H}$ maintain standing wave patterns according to $\sin \beta_{1z} z$ and $\beta_{1z} = \beta_1 \cos\theta_i$ where . No average power propagates along z $\vec{E}$ $\vec{H}$

as y component of and x component of are out of phase.

2. Along x i.e. parallel to the interface

y component of $\vec{E}$ and z component of $\vec{H}$ are in phase (both time and space) and propagate with phase velocity

$$v_{p1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1 \sin\theta_i}$$

$$\text{and } \lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta_i} \quad \text{.............................(6.78)}$$

The wave propagating along the x direction has its amplitude varying with z and hence constitutes a **non uniform** plane wave. Further, only electric field $\vec{E}_1$ is perpendicular to the direction of propagation (i.e. x), the magnetic field has component along the direction of propagation. Such waves are called transverse electric or TE waves.

ii. **Parallel Polarization**:

In this case also $\hat{a}_{ni}$ and $\hat{a}_{nr}$ are given by equations (6.69). Here $\vec{H}_1$ and $\vec{H}_r$ have only y component.

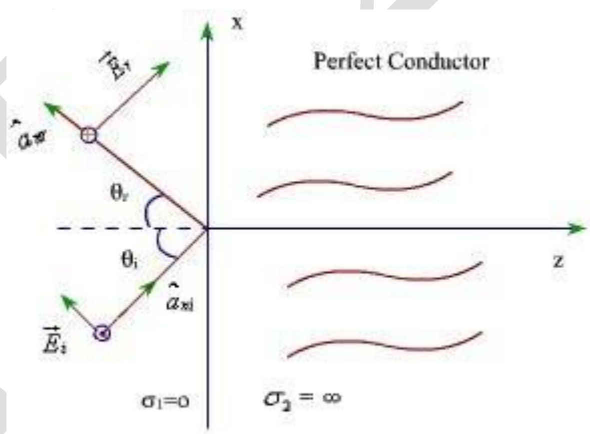


Figure 6.11: Parallel *Polarization*

With reference to fig (6.11), the field components can be written

as: Incident field components:

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right]e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{n_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
............................(6.79)

Reflected field components:

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right]e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{n_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
............................(6.80)

Since the total tangential electric field component at the interface is zero.

$$E_i(x,0) + E(x,0) = 0$$

Which leads to $E_{io} = -E_{ro}$ and $\theta_i = \theta_r$ as before.

Substituting these quantities in (6.79) and adding the incident and reflected electric and magnetic field components the total electric and magnetic fields can be written as

$$\vec{E}_i(x,z) = -2E_{io}\left[\hat{a}_x j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{a}_z \sin\theta_i \cos(\beta_1 z\cos\theta_i)\right]e^{-j\beta_1 x\sin\theta_i}$$

and $$\vec{H}_i(x,z) = \hat{a}_y \frac{2E_{io}}{n_1}\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}$$
...........................(6.81)

Once again, we find a standing wave pattern along z for the x and y components of $\vec{E}$ and $\vec{H}$, while a non uniform plane wave propagates along x with a phase velocity given $v_{p1x} = \frac{vp_1}{\sin\theta_i}$ by where $v_{p1} = \frac{\omega}{\beta_1}$. Since, for this propagating wave, magnetic field is in transverse direction, such waves are called transverse magnetic or TM waves.

Oblique incidence at a plane dielectric interface

We continue our discussion on the behavior of plane waves at an interface; this time we consider a plane dielectric interface. As earlier, we consider the two specific cases, namely parallel and perpendicular polarization.

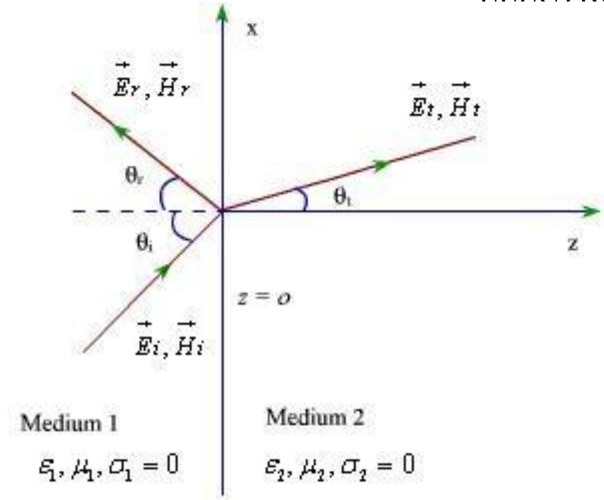


Fig 6.12: Oblique incidence at a plane dielectric interface

For the case of a plane dielectric interface, an incident wave will be reflected partially and transmitted partially.

In Fig(6.12), θ_i, θ_0 and θ_t corresponds respectively to the angle of incidence, reflection and transmission.

1. **Parallel Polarization**

As discussed previously, the incident and reflected field components can be written as

$$\vec{E}_i(x,z) = E_{io}\left[\cos\theta_i \hat{a}_x - \sin\theta_i \hat{a}_z\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

.... (6.82)

$$\vec{E}_r(x,z) = E_{ro}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

..........................(6.83)

In terms of the reflection coefficient Γ

$$\vec{E}_r(x,z) = \Gamma E_{io}\left[\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\hat{a}_y \frac{\Gamma E_{io}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

. ...(6.84)

The transmitted filed can be written in terms of the transmission coefficient T

$$\vec{E}_t(x,z) = TE_{io}\left[\hat{a}_x \cos\theta_t - \hat{a}_z \sin\theta_t\right] e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \hat{a}_y \frac{TE_{io}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

.........................(6.85)

We can now enforce the continuity of tangential field components at the boundary i.e. $z=0$

$$\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \Gamma\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = T\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } \frac{1}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{\Gamma}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{T}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$$

.........................(6.86)

If both E_x and H_y are to be continuous at $z=0$ for all x , then form the phase matching we have

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$\therefore$ We find that

$$\theta_i = \theta_r$$

$$\text{and } \beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$

.........................(6.87)

Further, from equations (6.86) and (6.87) we have

$$\cos\theta_i + \Gamma\cos\theta_i = T\cos\theta_t$$

$$\text{and } \frac{1}{\eta_1} - \frac{\Gamma}{\eta_1} = \frac{T}{\eta_2}$$

.........................(6.88)

$$\therefore \quad \cos\theta_i(1+\Gamma) = T\cos\theta_t$$

$$\text{and } \frac{1}{\eta_1}(1-\Gamma) = \frac{T}{\eta_2}$$

$$\therefore T = \frac{\eta_2}{\eta_1}(1-\Gamma)$$

$$\cos\theta_i(1+\Gamma) = \frac{n_2}{n_1}(1-\Gamma)\cos\theta_t$$

$$\therefore (n_1\cos\theta_i + n_2\cos\theta_t)\Gamma = n_2\cos\theta_t - n_1\cos\theta_i$$

or

$$\Gamma = \frac{n_2\cos\theta_t - n_1\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$

..........................(6.89)

$$\text{and } T = \frac{n_2}{n_1}(1-\Gamma)$$

$$= \frac{2n_2\cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i}$$

..........................(6.90)

From equation (6.90) we find that there exists specific angle $\theta_i = \theta_b$ for which $\Gamma = 0$ such that

$$n_2\cos\theta_t = n_1\cos\theta_b$$

or

$$\sqrt{1-\sin^2\theta_t} = \frac{n_1}{n_2}\sqrt{1-\sin^2\theta_b}$$

....(6.91)

Further,

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

$$\mu_1 = \mu_2 = \mu_0$$

..........................(6.92)

For non magnetic material

Using this condition

$$1-\sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}(1-\sin^2\theta_b)$$

$$\text{and } \sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_b$$

.........................(6.93)

From equation (6.93), solving for $\sin\theta_b$ we get

$$\sin\theta_b = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

This angle of incidence for which $\Gamma = 0$ is called Brewster angle. Since we are dealing with parallel polarization we represent this angle by $\theta_{b\|}$ so that

$$\sin\theta_{b\|} = \frac{1}{\sqrt{1+\frac{\varepsilon_1}{\varepsilon_2}}}$$

2. Perpendicular Polarization

For this case

$$\vec{E}_i(x,z) = \hat{a}_y E_{io} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{io}}{\eta_1}\left[-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{.........................(6.94)}$$

$$\vec{E}_r(x,z) = \hat{a}_y \Gamma E_{io} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = \frac{\Gamma E_{io}}{\eta_1}\left[\hat{a}_x\cos\theta_r + \hat{a}_z\sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad \text{.........................(6.95)}$$

$$\vec{E}_t(x,z) = \hat{a}_y T E_{io} e^{-j\beta_1(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = \frac{TE_{io}}{\eta_2}\left[-\hat{a}_x\cos\theta_t + \hat{a}_z\sin\theta_t\right] e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \quad \text{.........................(6.96)}$$

Using continuity of field components at *z=0*

$$e^{-j\beta_1 x\sin\theta_i} + \Gamma e^{-j\beta_1 x\sin\theta_r} = TE_{io} e^{-j\beta_2 x\sin\theta_t}$$

$$\text{and } -\frac{1}{\eta_1}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + \frac{\Gamma}{\eta_1}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = -\frac{T}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t} \quad \text{.........................(6.97)}$$

As in the previous case

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

$$\therefore \quad \theta_i = \theta_r$$

$$\text{and } \sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i \quad \text{.........................(6.98)}$$

Using these conditions we can write

$$1+\Gamma = T$$

$$-\frac{\cos\theta_i}{\eta_1} + \frac{\Gamma\cos\theta_i}{\eta_1} = -\frac{T\cos\theta_t}{\eta_2} \quad \text{.........................(6.99)}$$

From equation (6.99) the reflection and transmission coefficients for the perpendicular polarization can be computed as

$$\Gamma = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

$$\text{and} \quad T = \frac{2n_2 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} \quad \text{.........................(6.100)}$$

We observe that if $\Gamma = 0$ for an angle of incidence $\theta_i = \theta_b$

$$n_2 \cos\theta_b = n_1 \cos\theta_t$$

$$\therefore \cos^2\theta_t = \frac{n_2}{n_1}\cos^2\theta_b$$

$$= \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\cos^2\theta_b$$

$$\therefore 1 - \sin^2\theta_t = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\left(1 - \sin^2\theta_b\right)$$

Again $$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_b$$

$$\therefore \sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b$$

$$\therefore \left(1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_b\right) = \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\sin^2\theta_b$$

or $$\sin^2\theta_b\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2} - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right) = \left(1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or $$\sin^2\theta_b\left(\frac{\mu_1^2 - \mu_2^2}{\mu_1\mu_2\varepsilon_2}\right)\varepsilon_1 = \left(\frac{\mu_1\varepsilon_2 - \mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right)$$

or $$\sin^2\theta_b = \frac{\mu_2\left(\mu_1\varepsilon_2 - \mu_2\varepsilon_1\right)}{\varepsilon_1\left(\mu_1^2 - \mu_2^2\right)} \quad \text{........................ (6.101)}$$

We observe if $\mu_1 = \mu_2 = \mu_0$ i.e. in this case of non magnetic material Brewster angle does not exist as the denominator or equation (6.101) becomes zero. Thus for perpendicular polarization in dielectric media, there is Brewster angle so that Γ can be made equal to zero.

From our previous discussion we observe that for both polarizations

$$\sin\theta_t = \frac{\beta_1}{\beta_2}\sin\theta_i$$

If $\mu_1 = \mu_2 = \mu_0$

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i$$

For $\varepsilon_1 > \varepsilon_2$; $\theta_t > \theta_i$

The incidence angle $\theta_i = \theta_c$ for which $\theta_t = \frac{\pi}{2}$ i.e. $\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$ is called the critical angle of incidence.

:Electromagnetic spectrum:

If the angle of incidence is larger than θ_c total internal reflection occurs. For such case an evaneJITnt wave exists along the interface in the x direction (w.r.t. fig (6.12)) that attenuates exponentially in the normal i.e. z direction. Such waves are tightly bound to the interface and are called surface waves and waves spreading in the field of electric and magnetic together called electromagnetic spectrum.