MA8402 PROBABILITY AND QUEUING THEORY L T P C

4 0 0 4

OBJECTIVES:

- To provide necessary basic concepts in probability and random processes for applications

such as random signals, linear systems in communication engineering.

- To understand the basic concepts of probability, one and two dimensional random variables and to introduce some standard distributions applicable to engineering which can describe real life phenomenon.
- To understand the basic concepts of random processes which are widely used in IT fields.
- To understand the concept of correlation and spectral densities.
- To understand the significance of linear systems with random inputs.

UNIT I PROBABILITY AND RANDOM VARIABLES 12

Probability – Axioms of probability – Conditional probability – Baye‘s theorem - Discrete and continuous random variables – Moments – Moment generating functions – Binomial, Poisson, Geometric, Uniform, Exponential and Normal distributions.

UNIT II TWO - DIMENSIONAL RANDOM VARIABLES 12

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and linear regression – Transformation of random variables – Central limit theorem (for independent and identically distributed random variables).

UNIT III RANDOM PROCESSES12

Classification – Stationary process – Markov process - Markov chain - Poisson process – Random telegraph process.

UNIT IV QUEUEINGMODELS 12

Markovian queues – Birth and Death processes – Single and multiple server queueing models – Little‟s formula – Queues with finite waiting rooms – Queues with impatient customers: Balking and reneging.

UNIT V ADVANCED QUEUEING MODELS 12

Finite source models – M/G/1 queue – PollaczekKhinchin formula – M/D/1 and M/EK/1 as special cases – Series queues – Open Jackson networks.

TOTAL :60 PERIODS

OUTCOMES:

Upon successful completion of the course, students should be able to:

- Understand the fundamental knowledge of the concepts of probability and have knowledge of standard distributions which can describe real life phenomenon.
- Understand the basic concepts of one and two dimensional random variables and apply in engineering applications.
- Apply the concept random processes in engineering disciplines.
- Understand and apply the concept of correlation and spectral densities.
- The students will have an exposure of various distribution functions and help in acquiring skills in handling situations involving more than one variable. Able to analyze the response of random inputs to linear time invariant systems.

TEXT BOOKS:

1. Ibe, O.C.," Fundamentals of Applied Probability and Random Processes ", 1st Indian Reprint, Elsevier, 2007.

2. Peebles, P.Z., "Probability, Random Variables and Random Signal Principles ", Tata McGraw Hill, 4th Edition, New Delhi, 2002.

REFERENCES:

1. Cooper. G.R., McGillem. C.D., "Probabilistic Methods of Signal and System Analysis", Oxford University Press, New Delhi, 3rd Indian Edition, 2012.

2. Hwei Hsu, "Schaum‘s Outline of Theory and Problems of Probability, Random Variables and Random Processes ", Tata McGraw Hill Edition, New Delhi, 2004.

3. Miller. S.L. and Childers. D.G., ―Probability and Random Processes with Applications to Signal Processing and Communications ", Academic Press, 2004.

4. Stark. H. and Woods. J.W., ―Probability and Random Processes with Applications to Signal Processing ", Pearson Education, Asia, 3rd Edition, 2002.

5. Yates. R.D. and Goodman. D.J., ―Probability and Stochastic Processes", Wiley India Pvt. Ltd., Bangalore, 2nd Edition, 2012.

Subject Code:MA8402 Year/Semester: II /03
Subject Name: Probability &Queuing Theory Subject Handler: Dr. Shenbaga Ezhil

UNIT I –PROBABILITY & RANDOM VARIABLES
Probability – Axioms of probability – Conditional probability – Baye's theorem - Discrete and continuous random variables – Moments – Moment generating functions – Binomial, Poisson, Geometric, Uniform, Exponential and Normal distributions.

PART *A

Q.No.	Questions
1.	**Find the probability of a card drawn at random form an ordinary pack, is a diamond.** BTL2 Total number of ways of getting 1 card = 52 Number of ways of getting 1 diamond card is 13 $\Pr obability = \frac{Number\ of\ \ favourable\ events}{Number\ of\ exhaustive\ events}$ $= \frac{13}{52} = \frac{1}{4}$
2	**A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they both will be white.** BTL2 Total balls = 18 From these 18 balls 2 balls can be drawn in $18C_2$ ways Total number of ways of drawing 2 balls = 153 ---------------(1) 2 White balls can be drawn from 7 white balls in $7C_2$ ways. Therefore number of favourable cases = 21 Probability of drawing white balls $= \frac{No.,of\ \ favourable\ events}{Total\ no.,\ of\ cases}$ $= \frac{21}{153} = \frac{7}{51}$
3	**Write the axioms of probability**.BTL1 Let S be a sample space. To each event A, there is a real number P(A) satisfying the following axioms. (i) For any event A, $P(A) \geq 0$ (ii) P(S) = 1

	(iii) If $A_1, A_2, ..., A_n$ are finite number of disjoint events of S then $P(A_1 \cup A_2 \cup A_3 \cup ...) = P(A_1) + P(A_2) + P(A_3) + ...$
4	**A and B are events such that** $P(A \cup B) = \frac{3}{4}$; $P(A \cap B) = \frac{1}{4}, P(\overline{A}) = \frac{2}{3}$, *Find* $P(\overline{A}/B)$**.** BTL2 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $\frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$ $P(B) = \frac{2}{3}$ $P(\overline{A}/B) = \frac{P(\overline{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} = \frac{\frac{2}{3} - \frac{1}{4}}{\frac{2}{3}} = \frac{5}{8}$
5	**Define Baye's theorem.** BTL1 Let $A_1, A_2, ..., A_n$ be 'n' mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for I = 1,2,…n. Let 'B' be an event such that $B \subset \bigcup_{I=1}^{N} A_i$, $P(B) \neq 0$ then $P(A_i / B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^{n} P(A_i).P(B/A_i)}$
6	**Define Random variable. (Nov/Dec2013, Apr/May 2017)** BTL1 A random variable is a function that assigns a real number X(S) to every element $s \in S$ where 'S' is the sample space corresponding to a random experiment E.
7	**Prove that the function P(x) is a legitimate probability mass function of a discrete random variable X, where** $p(x) = \begin{cases} \frac{2}{3}\left(\frac{1}{3}\right)^x, & x = 0,1,2... \\ 0, & otherwise \end{cases}$ **(Apr/May 2017)** BTL5

	$\sum p(x) = \sum_{x=0}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^x = \frac{2}{3}\left(\frac{1}{3}\right)^0 + \frac{2}{3}\left(\frac{1}{3}\right)^1 + \frac{2}{3}\left(\frac{1}{3}\right)^2 + ...$ $= \frac{2}{3}\left[1+\frac{1}{3}+\left(\frac{1}{3}\right)^2 + ...\right]$ $= \frac{2}{3}\left[1-\frac{1}{3}\right]^{-1} = \frac{2}{3}\left[\frac{2}{3}\right]^{-1}$ $= \frac{2}{3}\left[\frac{3}{2}\right] = 1$ Since $\sum p(x) = 1$, the given function P(x) is a legitimate probability mass function of a discrete random variable 'X'.
8	**A random variable X has the following probability function.** (table below) **Find the value of 'a'.** BTL5 $\sum P(x) = 1$ $a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$ $81a = 1$ $a = \frac{1}{81}$
9	**If the random variable X takes the values 1,2,3 and 4nsuch that 2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]. Find the probability distribution (Nov/Dec 2016)**BTL3 Let P[X=3] = k 2P[X=1] = k $\Rightarrow p[X=1] = \frac{k}{2}$ 3P[X=2] = k $\Rightarrow p[X=2] = \frac{k}{3}$ 5P[X=4] = k $\Rightarrow p[X=4] = \frac{k}{5}$ We know that $\sum P(x) = 1$ $\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow \frac{61}{30}k = 1 \Rightarrow k = \frac{30}{61}$

X=x	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

The probability distribution of X is given by

X	1	2	3	4
P(x)	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

10

Find the variance of the discrete random variable X with the probability mass function $P_x(X)=\begin{cases}\frac{1}{3}; x=0\\ \frac{2}{3}; x=2\end{cases}$

(Nov/Dec2015 , Nov/Dec 2015)BTL3

The probability distribution of X given by

X	0	2
P(x)	$\frac{1}{3}$	$\frac{2}{3}$

$$E[X]=\sum x\,P(x)=(0)\left(\frac{1}{3}\right)+(2)\left(\frac{2}{3}\right)=0+\frac{4}{3}=\frac{4}{3}$$

$$E[X^2]=\sum x^2P(x)=(0)^2\left(\frac{1}{3}\right)+(2)^2\left(\frac{2}{3}\right)=\frac{8}{3}$$

$$VarX=E[X^2]-\left(E[X]\right)^2=\frac{8}{3}-\left(\frac{4}{3}\right)^2=\frac{8}{3}-\frac{16}{9}$$

11

Test whether the function defined as follows a density function ? $f(x)=\begin{cases}0 & x<2\\ \frac{1}{18}(3+2x) & 2\le x\le 4\\ 0 & x>4\end{cases}$ BTL4

$$\int_2^4 f(x)\,dx=\int_2^4\frac{1}{18}(3+2x)dx=\frac{1}{18}\left[3(x)_2^4+2\left(\frac{x^2}{2}\right)_2^4\right]$$

$$=\frac{1}{18}\left[3(4-2)+(16-4)\right]=\frac{1}{18}(18)=1$$

Hence the given function is a density function.

12

Show that the function $f(x)=\begin{cases}e^{-x} & x\ge 0\\ 0 & x<0\end{cases}$ **is a probability density function of a random variable X.**BTL5

	$\int f(x)\,dx = \int_0^\infty e^{-x}\,dx = \left[-e^{-x}\right]_0^\infty = -[0-1] = 1$ Hence the given function is a density function.
13	**Assume that X is a continuous random variable with the probability density function** $f(x)=\begin{cases}\frac{3}{4}\left(2x-x^2\right) & 0<x<2\\ 0 & otherwise\end{cases}$ **. Find P(X>1).** BTL3 $P[X>1]=\int_1^2 \frac{3}{4}\left(2x-x^2\right)dx = \frac{3}{4}\left[2\left(\frac{x^2}{2}\right)_1^2 - \left(\frac{x^3}{3}\right)_1^2\right]$ $= \frac{3}{4}\left[(4-1)-\left(\frac{8}{3}-\frac{1}{3}\right)\right]=\frac{1}{2}$
14	**A random variable X is known to have a distributive function** $F(x)=u(x)\left[1-e^{-x^2/b}\right], b>0$ **is a constant. Determine density function.** BTL 3 $f(x)=F_x(x)=\frac{d}{dx}\left[u(x)\left(1-e^{-x^2/b}\right)\right]$ $=u(x)\left(e^{-x^2/b}\left(-\frac{2x}{b}\right)\right)+u'(x)\left(1-e^{-x^2/b}\right)$ $=\frac{2}{b}xu(x)e^{-x^2/b}+u'(x)\left(1-e^{-x^2/b}\right)$
15	**If** $f(x)=\frac{x^2}{3}, -1<x<2$ **is the PDF of the random variable X then find P[0<X<1]. (Apr/May 2018)** BTL3 $\int f(x)\,dx=\int_0^1 \frac{x^2}{3}dx=\frac{1}{3}\left[\frac{x^3}{3}\right]_0^1=\frac{1}{9}[1-0]=\frac{1}{9}$
16	**A continuous random variable X has probability density function** $f(x)=\begin{cases}3x^2 & 0\le x\le 1\\ 0 & otherwise\end{cases}$ **Find 'k' such that P[X>k]=0.5** . BTL4

	P[X>k] = 0.5 $\Rightarrow \int_k^1 f(x)\,dx = 0.5$ $\Rightarrow \int_k^1 3x^2\,dx = 0.5$ $\Rightarrow 3\left[\frac{x^3}{3}\right]_k^1 = 0.5 \Rightarrow 1 - k^3 = 0.5$ $\Rightarrow k^3 = 1 - 0.5 = 0.5 \Rightarrow k = (0.5)^{\frac{1}{3}} = 0.7937$
17	**The cumulative distribution function of the random variable X is given by** $F_x(X) = \begin{cases} 0 & ; x < 0 \\ x + \frac{1}{2} & ; 0 \le x \le \frac{1}{2} \\ 1 & ; x > \frac{1}{2} \end{cases}$**. Find** $P[X > \frac{1}{4}]$. BTL3 $P\left[X > \frac{1}{4}\right] = 1 - P\left[X \le \frac{1}{4}\right] = 1 - F\left[\frac{1}{4}\right] = 1 - \left[\frac{1}{4} + \frac{1}{2}\right] = \frac{1}{4}$
18	**Find the moment generating function of Binomial distribution. (May/June 2013)**BTL3 The P.M.F of Binomial distribution is $P[X = x] = nC_x p^x q^{n-x}$, $x = 0,1,2,\ldots,n$ $M_x(t) = \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$ $= \sum_{x=0}^{n} nC_x q^{n-x} \left(pe^t\right)^x$ $= nC_0 q^{n-0}\left(pe^t\right)^0 + nC_1 q^{n-1}\left(pe^t\right)^1 + nC_2\, nC_0 q^{n-0}\left(pe^t\right)^0 q^{n-2}\left(pe^t\right)^2 + \ldots + nC_n q^{n-n}\left(pe^t\right)^n$ $= q^n + nC_1 q^{n-1}\left(pe^t\right) + nC_2 q^{n-2}\left(pe^t\right)^2 + \ldots + \left(pe^t\right)^n = \left(q + pe^t\right)^n$
19	**The mean & variance of Binomial distribution are 5 and 4. Determine the distribution.(Apr/May 2015)**BTL4 Given: Mean = np = 5, variance = npq =4

	$= 5q = 4 \Rightarrow q = \frac{4}{5}$ $p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$ $np = n\left(\frac{1}{5}\right) = 5 \Rightarrow n = 25$ The P.M.F of the binomial distribution is $P[X = x] = nC_x p^x q^{n-x} \quad x = 0,1,2, \ldots n$ $P[X = x] = 25C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{n-x}, \ x = 0,1,2,\ldots,25$
20	**Balls are tossed at random into 50 boxes. Find the expected number of tosses required to get the first ball in the fourth box. (Apr/May 2017)**BTL3 Let probability of success be $p = \frac{1}{50}$ According to Geometric distribution, Expected number of tosses to get the first ball in the fourth box = $E[x] = \frac{1}{p} = 50$
21.	**A random variable is uniformly distributed between 3 and 15. Find the variance of X. (Nov/Dec 2015)**BTL3 $Var\ X = \frac{(b-a)^2}{12}$ $= \frac{(15-3)^2}{12} = \frac{144}{12} = 12$
22.	**Messages arrive at a switchboard in a poisson manner at an average rate of six per hour. Find the probability for exactly 2 messages arrive within one hour. (Apr/May 2018)**BTL3 $Mean = \lambda = 6\ per\ hour$ $P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-6}6^x}{x!}$ $P[X = 2] = \frac{e^{-6}6^2}{2!} = 0.0446$
23.	**Find the moment generating function of Poisson distribution. (Nov/Dec 2014, Apr/May 2015)**BTL2

The P.M.F of Poisson distribution is

$$P[X=x]=\frac{e^{-\lambda}\lambda^{x}}{x!},\ x=0,1,2,\ldots \qquad \lambda>0$$

$$M_x(t)=E[e^{tx}]=\sum e^{tx}p(x)$$

$$=\sum_{x=0}^{\infty}e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!} \quad =e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{t})^{x}}{x!}$$

$$=e^{-\lambda}\left[1+\frac{(\lambda e^{t})^{1}}{1!}+\frac{(\lambda e^{t})^{2}}{2!}+\ldots\right]$$

$$=e^{-\lambda}e^{\lambda e^{t}}$$

24. **Let X be a random variable with M.G.F $M_x(t)=\frac{(2e^{t}+1)^{4}}{81}$. Find its mean and variance. (May/June 2016)**BTL3

$$M_x(t)=\frac{\left(1+2e^{t}\right)^{4}}{81}=\left(\frac{1+2e^{t}}{3}\right)^{4}=\left(\frac{1}{3}+\frac{2e^{t}}{3}\right)^{4}$$

Comparing the M.G.F of Binomial distribution , $M_x(t)=\left(q+pe^{t}\right)^{n}$, we have $p=\frac{2}{3},q=\frac{1}{3},n=4$

Hence

$$Mean=np=4\left(\frac{2}{3}\right)=\frac{8}{3}$$

$$Variance=npq=4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)=\frac{8}{9}$$

25. If **X and Y are independent random variables with variance 2 and 3. Find the variance of 3X+4Y. (May/June 2014)** BTL3

Given : Var(x)= 2 and Var(y) = 3
Var(aX+bY)= a^2Var(X) + b^2Var(Y)
Var(3X+4Y) = 9(2)+16(3)=66

26. **If** $f(x)=\begin{cases}cxe^{-x} & x>0\\ 0 & elsewhere\end{cases}$ **is the p.d.f of a random variable X. Find 'c'.**BTL5

$$\int_{0}^{\infty}cxe^{-x}\,dx=1$$

W.K.T $c\left[x\left(\frac{e^{-x}}{-1}\right)-(1)\left(e^{-x}\right)\right]_{0}^{\infty}=1$

$$c[(0)-(0-1)]=1$$

$$c=1$$

PART * B

1	**A random variable X has the following probability distribution**

X=x	-2	-1	0	1	2	3
P(X=x)	0.1	K	0.2	2k	0.3	3k

Find (i)The value of 'k'

(ii) Evaluate P(X>2) and P(-2<X<2)

(iii)Find the cumulative distributation of X

(**iv) Evaluate the mean of X(8M)(May/June 2010, Nov/Dec 2011, Nov/Dec 2017)**BTL5**.**

Answer:Page: 1.80-Dr.A. Singaravelu

- Total Probability $\sum P(x)=1$
- C.D. F $F(x)=P(X\le x)=\sum_{t\le x} p(t)$
- Mean $E(x)=\sum xP(x)$
- $E(x^2)=\sum x^2P(x)$
- $VarX=E(X^2)-[E(x)]^2$

- Using $\sum P(x)=1$, we have $k=\frac{1}{15}$. (1M)
- P(X<2)=0.5, $P(-2<X<2)=\frac{2}{5}$. (2M)
- C.D. F , F(-2)=0.1, F(-1)= 0.17, F(0)= 0.37, F(1)=0.5, F(2)= 0.8, F(3)=1. (3M)
- Mean E(x) = $\frac{16}{15}$. (2M)

2	**A random variable X has the following probability function**

X	0	1	2	3	4	5	6	7
P(x)	0	K	2k	2k	3k	K^2	$2k^2$	$7k^2+k$

Find (i) the value of 'k'

(ii) Evaluate $P[1.5<X<4.5/X>2]$

(iii) **The smallest value of** λ **for which** $P[X\le\lambda]>\frac{1}{2}$ **(8M)(Nov/Dec2012,May/June 2012, May/June 2014, A/M 2015)** BTL5

Answer:Page: 1.74-Dr.A.Singaravelu

- Total Probability $\sum P(x) = 1$
- C.D. F $F(x) = P(X \le x) = \sum_{t \le x} p(t)$
- Mean $E(x) = \sum xP(x)$
- $E(x^2) = \sum x^2 P(x)$
- $VarX = E(X^2) - [E(x)]^2$

- Value of $k = \frac{1}{10}$. (2M)
- $P[1.5 < X < 4.5 / X > 2] = \frac{P[1.5 < X < 4.5 \cap X > 2]}{P(X > 2)} = \frac{5}{7}$. (3M)
- The minimum value of $\lambda = 4$. (3M)

3

If the probability mass function of a random variable X is given by $P(X = r) = kr^3$ *r=1,2,3,4* Find the value of ‘k’, $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right)$, mean and variance of X. (8M)(Apr/May 2015) BTL5

Answer:Page: 1.24- Dr.G. Balaji

- Total Probability $\sum P(x) = 1$
- C.D. F $F(x) = P(X \le x) = \sum_{t \le x} p(t)$
- Mean $E(x) = \sum xP(x)$
- $E(x^2) = \sum x^2 P(x)$
- $VarX = E(X^2) - [E(x)]^2$

- Value of $k = \frac{1}{100}$. (2M)
- $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right) = \frac{P\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)}{P(X > 1)} = \frac{8}{99}$. (3M)
- Mean $E(X) = 3.54$, Var(X)= 0.4684. (3M)

4

If the moments of a random variable ‘X’ are defined by $E(X^r)$ =0.6; r=1,2,3,… Show that P(X=0)=0.4, P(X=1)=0.6, $P(X \ge 2) = 0$ BTL5

Answer: Page: 1.70-Dr.G. Balaji

- $M_x(t) = E\left(e^{tx}\right) = \sum_{x=0}^{\infty} e^{tx} p(x)$

	• $M_x(t) = \sum_{x=0}^{\infty} \frac{t^r}{r!} \mu_r{}'$ • $M_x(t) = \sum_{x=0}^{\infty} \frac{t^r}{r!} \mu_r{}' = 0.4 + (0.6)e^t$ • But $M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x) = p(0) + e^t p(1) + e^{2t} p(2)$. (3M) • Comparing P(X=0) = 0.4, P(X=1)=0.6. (3M) • $P(X \geq 2) = 0$. (2M)
5	**A continuous random variable X that can assume any value between x=2 and x=5 has a density function f(x) = k(1+x). Find P[X<4]. (8M) (Nov/Dec 2012, Apr/May 2015)** BTL5 **Answer: Page: 1.88- Dr.A.Singaravelu** • Total probabability $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_2^5 k(1+x)dx = 1$. (2M) • The value of $k = \frac{2}{27}$. (3M) • $P[X < 4] = \int_2^4 f(x)dx = \frac{16}{27}$. (3M)
6	**If the density function of a continuous random variable X is given by** $f(x) = \begin{cases} ax & ,0 \leq x \leq 1 \\ a & ,1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0 & ,otherwise \end{cases}$ **.Find the value of 'a, and find the c.d.f of X. (8M) (Apr/May 2015)**BTL5 **Answer :Page: 1.118- Dr. A. Singaravelu** • $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$ (1M) • Value of a=0.5. (1M) • For c.d.f , If x<0, F(x)=0. (1M) • If $0 \leq x \leq 1, F(x) = \frac{x^2}{4}$. (1M) • $1 \leq x \leq 2, F(x) = \frac{x}{2} - \frac{1}{4}$. (2M) • $2 \leq x \leq 3, F(x) = -\frac{x^2}{4} + \frac{3}{2}x - \frac{5}{4}$, For x>3, F(x)=1. (2M)

7	**A continuous random variable 'X' has the density function f(x) given by .** $f(x)=\frac{k}{1+x^2}, -\infty < x < \infty$ **Find the value of 'k' and the cumulative distribution of 'X'.(8M) (Nov/Dec 2014, Apr/May 2018)** BTL5 **Answer: Page: 1.123- Dr. A. Singaravelu** • $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{0}^{1} \frac{k}{1+x^2}dx = 1$. (2M) • The value of $k = \frac{1}{\pi}$. (2M) • The c.d.f is $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)dx = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$. (4M)
8	**Let 'X' be the random variable that denotes the outcome of the roll of a fair die. Compute the mean and variance of 'X'.(8M)(Apr/May 2018)** BTL4 **Answer : Page: 1.177- Dr. A. Singaravelu** • $P(X = i) = \frac{1}{6}, i = 1,2,...,6$. (1M) • $M_x(t) = \sum_{i=1}^{6} e^{it} P(X = i) = \frac{1}{6}\left[e^t + e^{2t} + ... + e^{6t}\right]$. (2M) • $E(x) = \left[M_x{}'(t)\right]_{t=0} = \frac{7}{2}$. (2M) • .$E(x^2) = \left[M_x{}''(t)\right]_{t=0} = \frac{91}{6}$. (2M) • $Var(X) = E(X^2) - \left[E(X)\right]^2 = \frac{35}{12}$. (1M)
9	**For the triangular distribution** $f(x)=\begin{cases} x & , 0 < x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0 & , otherwise \end{cases}$ **.Find the mean , variance, moment generating function. (8M) (Nov/Dec 2013)** BTL5 **Answer : Page: 1.180- Dr. A. Singaravelu** • $M_x(t) = E[e^{tx}] = \frac{\left[e^t - 1\right]^2}{t^2}$. (3M) • Mean $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = 1$. (2M) • $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx = \frac{7}{6}$. (2M) • $Var(X) = E(X^2) - \left[E(X)\right]^2 = \frac{1}{6}$ (1M)

10	**Find the M.G.F of the random variable X having the probability density function** $f(x)=\begin{cases}\frac{x}{4}e^{-x/2} & ,x>0\\ 0 & ,elsewhere\end{cases}$ **(8M) (May/June2012, May/June 2014)** BTL5 **Answer: Page:1.74-Dr. G. Balaji** • $M_x(t)=E[e^{tx}]=\int_0^\infty e^{tx}\frac{x}{4}e^{-x/2}\,dx=\frac{1}{(1-2t)^2}$. (1M) • $M_x(t)=1+\frac{t}{1!}\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+...$ (1M) • $M_x(t)=1+\frac{t}{1!}(4)+\frac{t^2}{2!}(24)+\frac{t^3}{3!}(192)+...$ (2M) • $\mu_1'=coefficient\ of\ \frac{t}{1!}=4$. (1M) • $\mu_2'=coefficient\ of\ \frac{t^2}{2!}=24$. (1M) • $\mu_3'=coefficient\ of\ \frac{t^3}{3!}=192$. (1M) • $\mu_4'=coefficient\ of\ \frac{t^4}{4!}=1920$. (1M)
11	**Find the MGF of the Binomial distribution and hence find the mean and variance. (8M)(Apr/May 2011, May/June2014)**BTL2 **Answer : Page: 1.190- Dr. A. Singaravelu** • $P(x)=nC_x p^x q^{n-x}$, $x=0,1,2,...,n$. (1M) • $M_x(t)=E[e^{tx}]=(q+pe^t)^n$. (2M) • Mean $E(X)=\left[M_x'(t)\right]_{t=0}=np$. (2M) • $E(X^2)=\left[M_x''(t)\right]_{t=0}=n^2p^2+npq$. (2M) • Var(X)= npq. (1M)
12	**Derive Poisson distribution form Binomial distribution. (8M)(Nov/Dec 2014, Nov/Dec 2017)**BTL2 **Answer : Page: 1.219 – Dr. A. Singaravelu** The Binomial distribution becomes Poisson distribution under the following conditions (2M) • The number of trials is very large • The probability of success is very small • $np=\lambda$

	• $P(X=x)=\lim_{n\to\infty} nC_x p^x q^{n-x} = \lim_{n\to\infty} \frac{(1-1/n)(1-2/n)\ldots(1-(x-1)/n)}{x!}\lambda^x \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$. (4M) • $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$. (2M)
13	**It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing atleast, exactly and atmost 2 defective items in a consignment of 1000packets using binomial and Poisson distribution.(8M) (Nov/Dec 2017)** BTL5 **Answer : Page: 1.116 – Dr. GBalaji** Probability of Binomial Distribution $P(X=x)=nC_x p^x q^{n-x}$ Probability of Poisson Distribution $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$ Binomial Distribution • Number of packets containing atleast 2 defective items $=NP(X\geq 2)=264$. (2M) • Number of packets containing exactly 2 defective items $=NP(X=2)=189$. (1M) • Number of packets containing atmost 2 defective items $=NP(X\leq 2)=925$. (1M) Poisson Distribution • Number of packets containing atleast 2 defective items $=NP(X\geq 2)=264$. (2M) • Number of packets containing exactly 2 defective items $=NP(X=2)=184$. (1M) • Number of packets containing atmost 2 defective items $=NP(X\leq 2)=920$. (1M)
14	**The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (1)without a breakdown, (2)with only one breakdown and (3)with atleast one breakdown(8M) (Nov/Dec 2017)** BTL5 **Answer : Page: 1.227- Dr. A. Singaravelu** Probability of Poisson Distribution $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$ • P(without a breakdown) = P(X=0) = 0.1653. (2M) • P(with only one breakdown) = P(X=1)=0.2975. (2M) • P(with atleast 1 breakdown)= $P(X\geq 1)=1-P(X<1)=0.8347$. (4M)
15	**State and prove the Memoryless property of Geometric distribution.(8M)(Nov/Dec2015, May/June 2016)** BTL1 **Answer : Page: 1.254- Dr. A. Singaravelu** Probability of Geometric distribution P(X=x) = $q^{x-1}p$, x=1,2,… • $P[X>m+n/X>m]=\frac{P[X>m+n\cap X>m]}{P[X>m]}$. (2M)

	• $P[X>k] = q^k$ (4M) • $P[X>m+n/X>m] = \frac{P[X>m+n]}{P[X>m]} = q^n$. (2M)
16	**If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8, what is the probability that he will finally pass the test (a) on the fourth trial , (b) in fewer than 4 trials. (8M) (May/June2015)** BTL5 **Answer : Page: 1.137- Dr. G. Balaji** Probability of Geometric distribution $P(X=x) = q^{x-1}p$, x=1,2,… • P(on the fourth trial) = P(X=4) = 0.0064. (4M) • P(fewer than 4 trials) = P(X<4) = 0.992. (4M)
17	**A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of a head occurring is 'p', find the value of 'p' so that the probability that an odd number of tosses is required, is equal to 0.6. Can you find a value of 'p' so that the probability is 0.5 that an odd number of tosses is required? (8M)(Nov/Dec 2010, Nov/Dec 2016)** BTL4 **Answer : Page: 1.135- Dr. G. Balaji** Probability of Geometric distribution $P(X=x) = q^{x-1}p$, x=1,2,… • P[X= odd number of tosses] $= \frac{1}{1+q} = 0.6$ (3M) • $q = \frac{2}{3}, p = 1-q = \frac{1}{3}$. (1M) • P[X= odd number of tosses] $= \frac{1}{1+q} = 0.5$ (3M) • q=1, p=0 . (1M)
18	**Determine the moment generating function of Uniform distribution in (a,b) and hence find the mean and variance. (8M) (Nov/Dec 2017, Apr/May 2018)**BTL2 **Answer : Page: 1.256-Dr. A. Singaravelu** The probability function of Uniform distribution is $f(x) = \begin{cases} \frac{1}{b-a}, a<x<b \\ 0 \quad , otherwise \end{cases}$ • $M_x(t) = E[e^{tx}] = \int_a^b e^{tx} f(x)dx = \frac{(e^{bt}-e^{at})}{t(b-a)}$. (3M) • Mean $E(X) = \int_a^b x\, f(x)\, dx = \frac{b+a}{2}$. (2M)

	• $E(X^2)=\int_a^b x^2\ f(x)\,dx=\frac{b^2+ab+a^2}{3}$. (2M) • $Var(X)=\frac{(b-a)^2}{12}$. (1M)
19	**Suppose 'X' has an exponential distribution with mean=10, Determine the value of 'x' such that P(X<x)=0.95. (8M) (Nov/Dec 2015, Apr/May 2017)**BTL5 **Answer : Page: 1.143- P. SivaramakrishnaDass** The probability function of exponential distribution is $f(x)=\begin{cases}\lambda e^{-\lambda x}, x\geq 0\\ 0 \quad ,otherwise\end{cases}$ • $Mean=\frac{1}{\lambda}=10 \Rightarrow \lambda=\frac{1}{10}$. (2M) • $P(X<x)= 1-P(X>x) =0.95$. (2M) • $1-e^{-\frac{x}{10}}=0.95 \Rightarrow x=29.96$. (4M)
20	**The time in hours required to repair a machine is exponentially distributed with perimeter** $\lambda=\frac{1}{2}$. (i) **What is the probability that the repair time exceeds 2h** (ii) **What is the conditional probability that a repair takes atleast 10h given that its duration exceeds 9h? (8M) (May/June 2012, Nov/Dec 2016, Nov/Dec 2017)**BTL3 **Answer : Page: 1.274- Dr. A. Singaravelu** The probability function of exponential distribution is $f(x)=\begin{cases}\lambda e^{-\lambda x}, x\geq 0\\ 0 \quad ,otherwise\end{cases}$ • P(the repair time exceeds 2h) $P(X>2)=\int_2^\infty \frac{1}{2}e^{-x/2}dx$ (2M) • $P(X>2)=0.3679$. (2M) • $P(X\geq 10/X>9)=P(X>1)=\int_1^\infty \frac{1}{2}e^{-x/2}dx$. (2M) • $P(X\geq 10/X>9)=0.6065$. (2M)
21	**In a test 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for (i)more than 2150 hours, (ii)less than 1950 hours and (iii) more than 1920 hours but less than 2160 hours. (8M) (Nov/Dec 2017)**BTL5 **Answer: Page:1.293 -A. Singaravelu** • $z=\frac{X-\mu}{\sigma}$

	• P(more than 2150 hrs) = P(X>2150) =P(z > 1.833) = 0.5-P(0 < z < 1.833) = 0.0336. (2M) • The number of bulbs expected to burn for more than 2150hrs = 2000 x 0.0336 = 67. (1M) • P(Less than 1950 hrs) = P(X< 1950) = P(z< -1.5) = 0.5 – P(0< z< 1.5) = 0.0668. (2M) • The number of bulbs expected to burn for less than 1950hrs = 2000 x 0.0668 = 134. (1M) • P(more than 1920 hrs but less than 2160 hrs) = P(1920 <X < 2160) = P(-2 < z < 2) = 0.9546.(1M) • The number of bulbs = 2000 x 0.9546 = 1909. (1M)
22	**In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution. (8M) (Nov/Dec 2012, Nov/Dec 2015)**BTL5 **Answer:Page: 1.295- A. Singaravelu** • $z = \frac{X-\mu}{\sigma}$ • $45-\mu = -0.49\sigma$. (2M) • P(Z > Z_1) = 0.8 or P(0 < Z < Z_2)= 0.42. (1M) • From tables , Z_2 = 1.40. (1M) • $64-\mu = 1.40\sigma$. (2M) • Solving, $\sigma = 10, \mu = 50$. (2M)
23	**The contents of urns I, II, III are as follows:** **1 white, 2 red and 3 black balls** **2 white, 3 red and 1 black balls and** **3 white, 1 red and 2 black balls.** **One urn is chosen at random and 2 balls are drawn. They happen to be white and red. What is the probability that they came from urns I, II, III.**BTL5 **Answer: Page: 1.60-Dr. A. Singaravelu** Let $A_1, A_2, ..., A_n$ be 'n' mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for I = 1,2,…n. Let 'B' be an event such that $B \subset \bigcup_{I=1}^{N} A_i$, $P(B) \neq 0$ then $P(A_i / B) = \frac{P(A_i).P(B/A_i)}{\sum_{i=1}^{n} P(A_i).P(B/A_i)}$ • P(E_1)=P(E_2)= P(E_3) = $\frac{1}{3}$ (1M) • $P(A/E_1) = \frac{1C_1 \times 2C_1}{6C_2} = \frac{2}{15}$, $P(A/E_2) = \frac{2C_1 \times 3C_1}{6C_2} = \frac{6}{15}$, $P(A/E_3) = \frac{3C_1 \times 1C_1}{6C_2} = \frac{3}{15}$ (2M) • $P(E_2/A) = \frac{P(E_2).P(A/E_2)}{\sum_{i=1}^{3} P(E_i).P(A/E_i)} = \frac{6}{11}$ (2M)

	• $P(E_3/A) = \frac{P(E_3).P(A/E_3)}{\sum_{i=1}^{3} P(E_i).P(A/E_i)} = \frac{3}{11}$ (2M) • $P(E_1/A) = 1 - P(E_2/A) - P(E_3/A) = \frac{2}{11}$ (1M)

UNIT II – TWO - DIMENSIONAL RANDOM VARIABLES

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and linear regression – Transformation of random variables – Central limit theorem (for independent and identically distributed random variables).

PART *A

Q.No.	Questions
1.	**State the basic properties of joint distribution of (X,Y) where X and Y are random variables. (May/June 2014)**BTL1 Properties of joint distribution of (X,Y) are (i) $F[-\infty, y] = 0 = F[x, -\infty]$ and $F[-\infty, -\infty] = 0, F[\infty, \infty] = 0$ (ii) $P[a < X < b, Y \le y] = F(b, y) - F(a, y)$ (iii) $P[X \le x, c < Y < d] = F(x, d) - F(x, c)$ (iv) $P[a < X < b, c < Y < d] = F(b, d) - F(a, d) - F(b, c) + F(a, c)$ (v) At points of continuity of f(x,y), $\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$
2	**The joint probability mass function of a two dimensional random variable (X,Y) is given by p(x,y) = f(2x + y); x = 1,2 and y = 1,2 where 'k' is a constant. Find the value of 'k'.(Nov/Dec 2015)**BTL5 The joint pmf of (X,Y) is <table><tr><td>x \ y</td><td>1</td><td>2</td></tr><tr><td>1</td><td>3k</td><td>4k</td></tr><tr><td>2</td><td>5k</td><td>6k</td></tr></table>We have $\sum\sum p(x, y) = 1$ Therefore, 3k + 4k + 5k + 6k = 1

	18 k=1 $k=\frac{1}{18}$.
3	**The joint probability density function of the random variables (X,Y) is given by** $f(x,y)=kxye^{-\left(x^2+y^2\right)},\ x>0, y>0$**. Find the value of 'k'. (Apr/May 2015)**BTL5 $\iint f(x,y)\,dkxdy=1$ $\int_0^\infty\int_0^\infty kxye^{-\left(x^2+y^2\right)}dx\,dy=1$ $k\int_0^\infty ye^{-y^2}dy\int_0^\infty xe^{-x^2}dx=1$ $\quad$ $Put\ x^2=t$ $2xdx=dt$ $k\int_0^\infty ye^{-y^2}dy\int_0^\infty e^{-t}\frac{dt}{2}=1$ $\quad$ $xdx=\frac{dt}{2}$ $\frac{k}{2}\int_0^\infty ye^{-y^2}\left[-e^{-t}\right]_0^\infty dy=1$ $\frac{k}{2}\int_0^\infty ye^{-y^2}\left[0+1\right]dy=1$ $\frac{k}{2}\int_0^\infty e^{-t}\frac{dt}{2}=1$ We have $\frac{k}{4}\left[-e^{-t}\right]_0^\infty=1$ $\frac{k}{4}[0+1]=1 \Rightarrow k=4$
4	**If the function f(x,y) = c(1-x)(1-y) , 0 < x < 1 , 0 < y < 1 is to be a density function, find the value of 'c'.(8M) (Nov/Dec 2017)**BTL5

	$\iint f(x,y)\,dx\,dy=1$ $\int_0^1\int_0^1 c(1-x)(1-y)\,dx\,dy=1$ $c\int_0^1(1-y)dy\int_0^1(1-x)dx=1$ $c\left[y-\frac{y^2}{2}\right]_0^1\left[x-\frac{x^2}{2}\right]_0^1=1$ $c\left[1-\frac{1}{2}\right]\left[1-\frac{1}{2}\right]=1$ $c\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]=1$ $c\left[\frac{1}{4}\right]=1 \Rightarrow c=4$
5	**The joint pdf of (X,Y) is** $f_{xy}(x,y)=xy^2+\frac{x^2}{8}, 0\le x\le 2, 0\le y\le 1$**. Find P(X<Y). (May/June 2013, Apr/May 2017)**BTL5 $P(X<Y)=\int_0^1\int_0^y\left(xy^2+\frac{x^2}{8}\right)dx\,dy$ $=\int_0^1\left[y^2\left(\frac{x^2}{2}\right)_0^y+\frac{1}{8}\left(\frac{x^3}{3}\right)_0^y\right]dy$ $=\int_0^1\left[\frac{y^2}{2}\left(y^2\right)+\frac{1}{24}\left(y^3\right)\right]dy=\int_0^1\left[\frac{y^4}{2}+\frac{y^3}{24}\right]dy$ $=\frac{1}{2}\left(\frac{y^5}{5}\right)_0^1+\frac{1}{24}\left(\frac{y^4}{4}\right)_0^1=\frac{1}{10}(1-0)+\frac{1}{96}(1-0)=\frac{53}{480}$
6	**If the joint pdf of (X,Y) is** $f(x,y)=\begin{cases}\frac{1}{4} & ,0<x,y<2\\ 0 & ,otherwise\end{cases}$ **. Find** $P[X+Y\le 1]$BTL5

	$P[X+Y\le 1]=\int_0^1\int_0^{1-y}\left(\frac{1}{4}\right)dx\,dy=\frac{1}{4}\int_0^1 (x)_0^{1-y}\,dy$ $=\frac{1}{4}\int_0^1 (1-y)dy \quad =\frac{1}{4}\left[y-\frac{y^2}{2}\right]_0^1$ $=\frac{1}{4}\left[1-\frac{1}{2}\right]=\frac{1}{8}$
7	**Find the marginal density function of X and Y if** $f(x,y)=\begin{cases}\frac{6}{5}\left(x+y^2\right) & ,0\le x,y\le 1\\ 0 & ,otherwise\end{cases}$ **(Nov/Dec 2012)**BTL5 Marginal density function of X is $f_x(x)=\int f(x,y)\,dy=\int_0^1\frac{6}{5}\left(x+y^2\right)dy=\frac{6}{5}\left[xy+\frac{y^3}{3}\right]_0^1=\frac{6}{5}\left[x+\frac{1}{3}\right]\ 0\le x\le 1$ Marginal density function of Y is $f_y(y)=\int f(x,y)\,dx=\int_0^1\frac{6}{5}\left(x+y^2\right)dy=\frac{6}{5}\left[\frac{x^2}{2}+y^2x\right]_0^1=\frac{6}{5}\left[\frac{1}{2}+y^2\right]\ 0\le y\le 1$
8	**The joint probability density function of the random variable X and Y is** $f(x,y)=\begin{cases}25e^{-5y} & ,\ 0<x<0.2,\ y>0\\ 0 & ,otherwise\end{cases}$ **.Find the marginal PDF of X and Y. (Nov/Dec 2016)**BTL5 Marginal density function of X is $f_x(x)=\int f(x,y)\,dy=\int_0^\infty 25e^{-5y}dy=25\left[\frac{e^{-5y}}{-5}\right]_0^\infty=-5[0-1]=5\ \ 0\le x\le 0.2$ Marginal density function of Y is $f_y(y)=\int f(x,y)\,dx=\int_0^{0.2}25e^{-5y}dx=25e^{-5y}\left[x\right]_0^{0.2}=2e^{-5y}\left[0.2-0\right]=5e^{-5y}\quad y>0$
9	**If X and Y are independent random variables having the joint density function** $f(x,y)=\frac{1}{8}(6-x-y),0<x<2,\ 2<y<4$**. Find P[X+Y<3]**. BTL5

	$P[X+Y<3]=\frac{1}{8}\int_2^3\int_0^{3-y}(6-x-y)\,dx\,dy$ $=\frac{1}{8}\int_2^3\left[(6-y)(x)-\frac{x^2}{2}\right]_0^{3-y}dy=\frac{1}{8}\int_2^3\left[(6-y)(3-y)-\frac{(3-y)^2}{2}\right]dy$ $=\frac{1}{8}\int_2^3\left[18-9y+y^2-\frac{1}{2}(3-y)^2\right]dy$ $=\left[18y-9\frac{y^2}{2}+\frac{y^3}{3}-\frac{1}{2}\frac{(3-y)^3}{-3}\right]_2^3$ $=\left[18(3)-\frac{9}{2}(9)+\frac{27}{3}+\frac{1}{6}(0)\right]-\left[18(2)-\frac{9}{2}(4)+\frac{8}{3}+\frac{1}{6}(1)\right]$ $=\left[18-\frac{45}{2}+\frac{19}{3}-\frac{1}{6}\right]=\frac{5}{24}$
10	**Let X and Y be random variables with joint density function** $f(x,y)=\begin{cases}4xy & ,0\le x\le 1,\ 0\le y\le 1\\ 0 & ,otherwise\end{cases}$ **Find E[XY].**BTL5 $E[XY]=\int\int xy\,f(x,y)\,dx\,dy=\int_0^1\int_0^1 xy(4xy)\,dx\,dy$ $=4\int_0^1 x^2dx\int_0^1 y^2dy$ $=4\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^3}{3}\right]_0^1=\frac{4}{9}(1)(1)=\frac{4}{9}$
11	**Let X and Y be a two-dimensional random variable. Define covariance of (X,Y). If X and Y are independent, what will be the covariance of (X,Y)? (May/June 2016)**BTL2 Covariance of (X,Y) is defined as *Cov(X,Y) = E[XY] – E[x]E[Y]* If X and Y are independent, then *Cov(X,Y) =0.*
12	**Two random variables X and Y have the joint pdf** $f(x,y)=\begin{cases}\frac{xy}{96} & ;0<x<4,1<y<5\\ 0 & ;otherwise\end{cases}$ **. Find** ***Cov(X,Y)*. (May/June 2016)**BTL5 *Cov(X,Y) = E[XY] – E[x]E[Y]*

	$E[X]=\int\int x\, f(x,y)\,dx\,dy=\int_1^5\int_0^4 x\left(\frac{xy}{96}\right)dx\,dy=\frac{1}{96}\int_1^5 y\,dy\int_0^4 x^2\,dx$ $=\frac{1}{96}\left[\frac{y^2}{2}\right]_1^5\left[\frac{x^3}{3}\right]_0^4=\frac{1}{576}[25-1][64]=\frac{8}{3}$ $E[Y]=\int\int y\, f(x,y)\,dx\,dy=\int_1^5\int_0^4 y\left(\frac{xy}{96}\right)dx\,dy=\frac{1}{96}\int_1^5 y^2\,dy\int_0^4 x\,dx$ $==\frac{1}{96}\left[\frac{y^3}{3}\right]_1^5\left[\frac{x^2}{2}\right]_0^4=\frac{1}{576}[125-1][16]=\frac{31}{9}$ $E[XY]=\int\int xy\, f(x,y)\,dx\,dy=\int_1^5\int_0^4 xy\left(\frac{xy}{96}\right)dx\,dy=\frac{1}{96}\int_1^5 y^2\,dy\int_0^4 x^2\,dx$ $=\frac{1}{96}\left[\frac{y^3}{3}\right]_1^5\left[\frac{x^3}{3}\right]_0^4=\frac{1}{864}[125-1][64]=\frac{248}{27}$ $\therefore\ Cov(X,Y)=\left[\frac{248}{27}\right]-\left[\frac{8}{3}\right]\left[\frac{31}{9}\right]=0$
13	**Let X and Y be any two random variables a,b be constants. Prove that Cov(aX,bY)=abCov(X,Y).**BTL5 *Cov(X,Y) = E[XY] –E[X]E[Y]* *Cov(aX,bY)= E[aXbY] – E[aX] E[bY]* =ab E[XY] –ab E[X]E[Y] =ab[*E[XY] –E[X]E[Y]]* = ab*Cov(X,Y)*
14	**If Y = - 2X + 3, Find Cov(X,Y).**BTL3 *Cov(X,Y) = E[XY] –E[X]E[Y]* =E[X(-2X+3)]-E[X]E[-2X+3] = E[$-2X^2$+3X] – E[X][-2E[X]+3] = -2E[X^2]+3E[X]+2$(E[X])^2$-3E[x] = -2(E[X^2]-$(E[X])^2$) = -2Var X
15	**If X_1 has mean 4 and variance 9 while X_2 has mean -2 and variance 5 and the two are independent , find Var($2X_1+X_2-5$).**BTL3 E[X_1]=4 , E[X_2] = -2 Var[X_1] = 9, Var[X_2] = 5 Var($2X_1+X_2-5$) = 4 VarX_1 + VarX_2

	$= 4(9) + 5 = 41.$
16	**If X and Y are independent random variables then show that E[Y/X] = E[Y] ,E[X/Y] = E[X]. (Nov/Dec 2016)**BTL5 $E[Y/X] = \int y.\frac{f(x,y)}{f(x)}dy$ Since X and Y are independent, $E[Y/X] = \int y.\frac{f(x).f(y)}{f(x)}dy = \int y.f(y)dy = E[Y]$ $E[X/Y] = \int x.\frac{f(x,y)}{f(y)}dx$ Since X and Y are independent, $E[X/Y] = \int x.\frac{f(x).f(y)}{f(y)}dx = \int x.f(x)dx = E[X]$
17	**Find the acute angle between the two lines of regression. (Apr/May 2015, Apr/May 2018)**BTL3 The equations of the regression are $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \quad ----------(1)$ $x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \quad ----------(2)$ Slope of line (1) is $m_1 = r\frac{\sigma_y}{\sigma_x}$ Slope of line (2) is $m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$ If θ is the acute angle between the two lines, then $\tan\theta = \frac{\lvert m_1 - m_2\rvert}{1 + m_1 m_2}$ $= \frac{\left\lvert r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}\right\rvert}{1 + r\frac{\sigma_y}{\sigma_x}.\frac{\sigma_y}{r\sigma_x}} = \frac{\left\lvert\frac{(r^2-1)}{r}\frac{\sigma_y}{\sigma_x}\right\rvert}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$ $= \frac{\left\lvert\frac{-(1-r^2)}{r}\frac{\sigma_y}{\sigma_x}\right\rvert}{\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}} = \frac{(1-r^2)\sigma_x\sigma_y}{\lvert r\rvert(\sigma_x^2 + \sigma_y^2)}$
18	**The regression equations are 3x + 2y = 26 and 6x + y = 31. Find the correlation coefficient between X and Y.** BTL5 Let 3x + 2y = 26 be the regression equation of Y on X.

	Therefore, $2y = -3x + 26 \Rightarrow y = -\frac{3}{2}x + \frac{26}{2}$. The regression coefficient $b_{yx} = -\frac{3}{2}$ Let 6x+ y = 31 be the regression equation of X on Y. Therefore, $6x = -y + 31 \Rightarrow x = -\frac{1}{6}y + \frac{31}{6}$ The regression coefficient $b_{xy} = -\frac{1}{6}$ Hence, correlation coefficient r_{xy} is given by $r_{xy} = \pm\sqrt{b_{yx} \times b_{xy}} = \pm\sqrt{\left(\frac{-3}{2}\right)\left(\frac{-1}{6}\right)} = \pm\sqrt{\frac{1}{4}} = \pm 0.5$ =-0.5, since both the regression coefficients are negative.
19	**The two regression equations of two random variables X and Y are 4x – 5y +33 = 0 and 20x – 9y =107. Find the mean values of X and Y. (Nov/Dec 2015)** BTL5 Replace x and y as $\bar{x}$ and $\bar{y}$, we have $4\bar{x} - 5\bar{y} = -33 \quad ------(1)$ $20\bar{x} - 9\bar{y} = 107 \quad ------(2)$ Solving the equations (1) and (2), we have $\bar{x} = 13$ and $\bar{y} = 17$.
20	**Can y=5+2.8x and x=3-0.5y be the estimated regression equations of y on x and x on y respectively, explain your answer. (Nov/Dec 2016)**BTL4 Since the signs of regression co-effieients are not the same, the given equation is not estimated regression equation of y on x and x on y.
21	**If X has an exponential distribution with parameter 1. Find the pdf of $y = \sqrt{x}$.**BTL3 $y = \sqrt{x} \Rightarrow x = y^2$ Since $dx = 2y\,dy \Rightarrow \frac{dx}{dy} = 2y$ Since X has an exponential distribution with parameter 1, the pdf of X is given by, $f_x(x) = e^{-x}, x > 0 \qquad [f(x) = \lambda e^{-\lambda x}, \lambda = 1]$ $\therefore f_y(y) = f_x(x)\left\|\frac{dx}{dy}\right\|$ $= e^{-x} 2y = 2ye^{-y^2} \quad y > 0$
22	**State Central limit theorem.**BTL1 If X_1, X_2, …,X_n,… be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, i=1,2,… and if $S_n = X_1 + X_2 + \ldots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$
23	**If X and Y have joint pdf of** $f(x,y) = \begin{cases} x+y, & 0 < x, y < 1 \\ 0 & , elsewhere \end{cases}$**. Check whether X and Y are independent.**BTL4 The marginal function of X is

$$f(x)=\int_0^1 (x+y)dy=\left[xy+\frac{y^2}{2}\right]_0^1=x+\frac{1}{2},\ 0<x<1$$

The marginal function of Y is

$$f(y)=\int_0^1 (x+y)dx=\left[\frac{x^2}{2}+y\right]_0^1=y+\frac{1}{2},\ 0<y<1$$

Now, $f(x).f(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)=xy+\frac{1}{2}(x+y)+\frac{1}{4}\neq x+y\neq f(x,y)$

Hence X and Y are not independent.

24

Assume that the random variables X and Y have the probability density function f(x,y). What is E[E[X/Y]]? (Apr/May 2017)BTL5

$$E\big[\left[X/Y\right]\big]=\int_{-\infty}^{\infty} E\left[X/Y\right] f(y)\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x/y)\,dx\, f(y)\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x/y)\ f(y)dx\,dy$$

$$=\int_{-\infty}^{\infty} x\int_{-\infty}^{\infty}\ f(x,y)\,dy\,dy\,dx$$

$$=\int_{-\infty}^{\infty} x\, f(x)\,dx=E(X)$$

25

Define the joint density function of two random variables X and Y. BTL1

If (X,Y) is a two dimensional continuous random variables such that $P\left[x-\frac{dx}{2}\le X\le x+\frac{dx}{2}, y-\frac{dy}{2}\le Y\le y+\frac{dy}{2}\right]=f(x,y)dx\,dy$, then f(x,y) is called the joint pdf of (X,Y), provided f(x,y) satisfies the following conditions

(i) $f(x,y)\ge 0,\ for\ all\ (x,y)\in R$

(ii) $\iint_R f(x,y)dx\,dy=1$

Part*B

1

The joint pmf of (X,Y) is given by P(x,y) = k(2x + 3y), x = 0,1,2 ; y = 1,2,3. Find all the marginal and conditional probability distributions. Also, find the probability distribution of (X+Y). (10M) (Nov/Dec 2014, Nov/Dec 2015) BTL5

Answer: Pg. 2.8 – Dr. A. Singaravelu

- $k=\frac{1}{72}$. (1M)
- Marginal distribution of X: $P(X=0)=\frac{18}{72}, P(X=1)=\frac{24}{72}, P(X=2)=\frac{30}{72}$ (1M)
- Marginal distribution of Y: $P(Y=1)=\frac{15}{72}, P(Y=2)=\frac{24}{72}, P(Y=3)=\frac{33}{72}$ (1M)

	• Conditional distribution of X given Y: $P\left[X = x_i / Y = y_1\right] = \frac{1}{5}, \frac{1}{3}, \frac{7}{15}$ (1M) • $P\left[X = x_i / Y = y_2\right] = \frac{1}{4}, \frac{1}{3}, \frac{5}{12}$. (1M) • $P\left[X = x_i / Y = y_3\right] = \frac{9}{33}, \frac{1}{3}, \frac{13}{33}$. (1M) • Conditional distribution of Y given X: $P\left[Y = y_i / X = x_0\right] = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. (1M) • $P\left[Y = y_i / X = x_1\right] = \frac{5}{24}, \frac{1}{3}, \frac{11}{24}$. (1M) • $P\left[Y = y_i / X = x_2\right] = \frac{7}{30}, \frac{1}{3}, \frac{13}{30}$. (1M) • Total probability distribution of X+Y is 1. (1M)
2	**The two dimensional random variable (X,Y) has the joint pmf** $f(x,y) = \frac{x+2y}{27}, x = 0,1,2; y = 0,1,2$ **Find the conditional distribution of Y for X=x. (8M) (Nov/Dec 2017)** BTL5 **Answer : Pg. 2.13 – Dr. A. Singaravelu** • Marginal distribution of X: $P(X=0) = \frac{6}{27}, P(X=1) = \frac{9}{27}, P(X=2) = \frac{12}{27}$ (1M) • Marginal distribution of Y: $P(Y=0) = \frac{3}{27}, P(Y=1) = \frac{9}{27}, P(Y=2) = \frac{15}{27}$ (1M) • Conditional distribution of Y given X: $P\left[Y = y_i / X = x_0\right] = 0, \frac{1}{3}, \frac{2}{3}$. (2M) • $P\left[Y = y_i / X = x_1\right] = \frac{1}{9}, \frac{1}{3}, \frac{5}{9}$. (2M) • $P\left[Y = y_i / X = x_2\right] = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. (2M)
3	**Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denote the number of red balls drawn, find the joint probability distribution of (X,Y).(8M)(Apr/May 2015, May/June 2016)** BTL5 **Answer: Page: 2.20- Dr. G. Balaji** • Let X denote number of white balls drawn and Y denote the number of red balls drawn. • $P(X=0,Y=0) = \frac{1}{21}$, $P(X=0,Y=1) = \frac{3}{14}, P(X=0,Y=2) = \frac{1}{7}, P(X=0,Y=3) = \frac{1}{84}$ (3M) • $P(X=1,Y=0) = \frac{1}{7}$, $P(X=1,Y=1) = \frac{2}{7}, P(X=1,Y=2) = \frac{1}{14}$ (3M) • $P(X=2,Y=0) = \frac{1}{21}$, $P(X=2,Y=1) = \frac{1}{28}$ (2M)
4	**The joint pdf of the random variable (X,Y) is given by** $f(x,y) = Kxye^{-(x^2+y^2)}, x>0, y>0$. **Find the value of 'K' and also prove that X and Y are independent. (8M) (Apr/May 2015)**BTL5 **Answer : Pg. 2.25 – Dr.A. Singaravelu**

- Marginal density function of X : $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$
- Marginal density function ofY: $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$
- X and Y are independent if f(x,y) = f(x). f(y)

- $\int_0^{\infty}\int_0^{\infty} Kxye^{=\left(x^2+y^2\right)}dx\,dy = 1 \Rightarrow K = 4.$ (2M)
- Marginal density function of X : $f(x) = \int_0^{\infty} Kxye^{=\left(x^2+y^2\right)}dy = 2xe^{-x^2}$. (2M)
- Marginal density function of Y : $f(y) = \int_0^{\infty} Kxye^{=\left(x^2+y^2\right)}dx = 2ye^{-y^2}$. (2M)
- f(x). f(y) = $2xe^{-x^2}.2ye^{-y^2} = 4xye^{=\left(x^2+y^2\right)} = f(x,y).$ (2M)

5 **Given** $f_{XY}(x,y) = Cx(x-y), 0 < x < 2, -x < y < x$ **and 0 elsewhere. (a)Evaluate C; (b)Find $f_x(x)$; (c)** $f_{y/x}\left(\frac{y}{x}\right)$ **(d)**Find $f_y(y)$.**(8M) (May, June 2013May/June2016)**BTL5

Answer : Pg. 2.40 – Dr. A. Singaravelu

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

- Marginal density function of X : $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$
- Marginal density function ofY: $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

- $\int_0^2\int_{-x}^{x} Cx(x-y)\,dy\,dx = 1 \Rightarrow C = \frac{1}{8}$. (1M)
- $f_x(x) = \int_{-x}^{x} Cx(x-y)\,dy = \frac{x^3}{4}, 0 < x < 2.$ (2M)
- $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{x-y}{2x^2}, -x < y < x.$ (2M)
- $f_y(y) = \begin{cases} \int_{-y}^{2}\frac{1}{8}x(x-y)dx, if\ -2 \le y \le 0 \ = \frac{1}{3} - \frac{y}{4} + \frac{5}{28}y^3 \\ \int_{y}^{2}\frac{1}{8}x(x-y)dx, if\ 0 \le y \le 2 \ = \frac{1}{3} - \frac{y}{4} + \frac{1}{28}y^3 \end{cases}$ (3M)

6	**The joint pdf of (X,Y) is given by $f(x,y)=e^{-(x+y)}, 0 \le x, y \le \infty$. Are X and Y independent.(8M)(Nov/Dec 2015, Apr/May 2018)** BTL4 **Answer :Page:2.28 – Dr. A. Singaravelu** • Marginal density function of X : $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$ • Marginal density function ofY: $f(y)=\int_{-\infty}^{\infty} f(x,y)dx$ • X and Y are independent if f(x,y) = f(x). f(y) • $f(x)=\int_0^{\infty} e^{-(x+y)}dy=e^{-x}$. (3M) • $f(y)=\int_0^{\infty} e^{-(x+y)}dx=e^{-y}$. (3M) • $f(x).f(y)=e^{-x}e^{-y}=e^{-(x+y)}=f(x,y)$. (2M)
7	**The joint p.d.f of a two dimensional random variable (X,Y) is given by $f(x,y)=xy^2+\frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$. Compute (i) $P\left(X>1/Y<\frac{1}{2}\right)$, (ii) $P\left(Y<\frac{1}{2}/X>1\right)$, (iii) P(X<Y), (iv) $P(X+Y \le 1)$(8M) (Apr/May 2017)** BTL5 **Answer : Pg. 2.43 – Dr.A. Singaravelu** • $P\left(X>1/Y<\frac{1}{2}\right)=\frac{P\left(X>1, Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)}=\frac{\frac{5}{24}}{\frac{1}{4}}=\frac{5}{6}$ (2M) • $P\left(Y<\frac{1}{2}/X>1\right)=\frac{P\left(X>1, Y<\frac{1}{2}\right)}{P(X>1)}=\frac{\frac{5}{24}}{\frac{19}{24}}=\frac{5}{19}$ (2M) • $P(X<Y)=\int_0^1\int_0^y\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\frac{53}{480}$ (2M) • $P(X+Y \le 1)=\int_0^1\int_0^{1-y}\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\frac{13}{480}$ (2M)
8	**Let X and Y have j.d.f f(x,y) = k , 0<x<y<2, Find the marginal pdf. Find the conditional density functions.(8M) (Nov/Dec 2016, Nov/Dec 2017)** BTL5 **Answer : Pg. 2.33 – Dr. A. Singaravelu** • $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy=1$ • Marginal density function of X : $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$

- Marginal density function ofY: $f(y)=\int_{-\infty}^{\infty} f(x,y)dx$
- The conditional density function of X given Y: $f(X/Y)=\dfrac{f(x,y)}{f(y)}$
- The conditional density function of Y given X: $f(Y/X)=\dfrac{f(x,y)}{f(x)}$

- $\int_0^2\int_0^y k\,dx\,dy=1 \Rightarrow k=\frac{1}{2}.$ (2M)
- $f(x)=\int_x^2 \frac{1}{2}dy=\frac{1}{2}(2-x), 0<x<1$ (2M)
- $f(y)=\int_0^y \frac{1}{2}dx=\frac{y}{2}, 0<y<2$ (2M)
- $f(X/Y)=\frac{1}{y}, 0<x<y$ (1M)
- $f(Y/X)=\frac{1}{2-x}, x<y<2$ (1M)

9

If the joint distribution function of X and Y is given by $F(x,y)=(1-e^{-x})(1-e^{-y}), x>0, y>0$. Find the marginal density function of X and Y. Check if X and Y are independent. Also find P(1<X<3 / 1<Y<2). (8M) (Apr/May 2015, May/June 2016) BTL5

Answer :Pg. 2.50 – Dr. A. Singaravelu

- $f(x,y)=\dfrac{\partial^2 F(x,y)}{\partial x \partial y}=e^{-(x+y)}$
- $f(x)=\int_0^{\infty} e^{-(x+y)}dy=e^{-x}.$ (2M)
- $f(y)=\int_0^{\infty} e^{-(x+y)}dx=e^{-y}.$ (2M)
- $f(x).f(y)=e^{-x}e^{-y}=e^{-(x+y)}=f(x,y).$ (2M)
- $P(1<X<3, 1<Y<2)=\left(\dfrac{1-e^2}{e^3}\right)\left(\dfrac{1-e}{e^2}\right).$ (2M)

10

Find the co-efficient of correlation between X and Y from the data given below.(8M) (May 2016) BTL5

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

Answer : Page: 2.71- Dr. A. Singaravelu

- $\overline{X}=\dfrac{\sum X}{n}=\dfrac{544}{8}=68$ (1M)

	• $\bar{Y} = \frac{\sum Y}{n} = \frac{552}{8} = 69$ (1M) • $\sigma_x = \sqrt{\frac{1}{n}\sum X^2 - \bar{X}^2} = 2.121$ (2M) • $\sigma_y = \sqrt{\frac{1}{n}\sum Y^2 - Y^2} = 2.345$ (2M) • $r(X,Y) = \frac{Cov(X,Y)}{\sigma_x . \sigma_y} = 0.6031$ (2M)
11	**Let X and Y be discrete random variables with pdf** $f(x,y) = \frac{x+y}{21}, x = 1,2,3; y = 1,2$ **. Find** $\rho(X,Y)$ **(8M)** BTL5 **Answer : Pg. 2.78- Dr. A. Singaravelu** • $E(X) = \sum x\, f(x) = \frac{46}{21}$ (1M) • $E(Y) = \sum y\, f(y) = \frac{33}{21}$ (1M) • $E(X^2) = \sum x^2\, f(x) = \frac{114}{21}$ (1M) • $E(Y^2) = \sum y^2\, f(y) = \frac{57}{21}$ (1M) • $Var\, X = \sigma_x^2 = E(X^2) - [E(X)]^2 = \frac{278}{441}$ (1M) • $Var\, Y = \sigma_y^2 = E(Y^2) - [E(Y)]^2 = \frac{108}{441}$ (1M) • $E(XY) = \sum xy\, f(x,y) = \frac{72}{21}$ (1M) • $r(X,Y) = \frac{Cov(X,Y)}{\sigma_x . \sigma_y} = \frac{-6}{173.20} = -0.035$ (1M)
12	**If the joint pdf of (X,Y) is given by** $f(x,y) = x+y, 0 \le x, y \le 1$**. Find** ρ_{xy} **.(8 M) (May/June 2014)** BTL3 **Answer : Page : 2.99 – Dr. A. Singaravelu** • $f(x) = \int_0^1 (x+y)dy = x + \frac{1}{2}, 0 < x < 1$ (1M) • $f(y) = \int_0^1 (x+y)dx = y + \frac{1}{2}, 0 < y < 1$ (1M) • $E(X) = \int x\, f(x)dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \frac{7}{12}$ (1M) • $E(Y) = \int y\, f(y)dy = \int_0^1 y\left(y + \frac{1}{2}\right)dy = \frac{7}{12}$ (1M)

	• $E(X^2)=\int x^2 f(x)dx=\frac{5}{12}$, $E(Y^2)=\int y^2 f(y)dy=\frac{5}{12}$ (1M) • $Var\,X=\sigma_x^2=E(X^2)-[E(X)]^2=\frac{11}{144}$, $Var\,Y=\sigma_y^2=E(Y^2)-[E(Y)]^2=\frac{11}{144}$ (1M) • Cov(X,Y) = E(XY) – E(X). E(Y) = $\frac{-1}{144}$ (1M) • $r(X,Y)=\frac{Cov(X,Y)}{\sigma_x.\sigma_y}=\frac{-1}{11}$ (1M)
13	**Two independent random variables X and Y are defined by,** $f(x)=\begin{cases}4ax, 0\le x\le 1\\ 0 \quad ,otherwise\end{cases}$ $f(y)=\begin{cases}4by, 0\le y\le 1\\ 0 \quad ,otherwise\end{cases}$**. Show that U=X + Y and V=X – Y are uncorrelated. (8 M)(May/June 2013)** BTL4 **Answer : Page: 2.105 – Dr. A. Singaravelu** • $\int_0^1 f(x)dx=1 \Rightarrow a=\frac{1}{2}$; $\int_0^1 f(y)dy=1 \Rightarrow b=\frac{1}{2}$ (1M) • $E(U)=E(X)+E(Y)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$. (2M) • $E(V)=E(X)-E(Y)=\frac{2}{3}-\frac{2}{3}=0$. (2M) • $E(UV)=E(X^2)-E(Y^2)=\frac{1}{2}-\frac{1}{2}=0$. (2M) • Cov(U,V) = E(UV) – E(U).E(V) =0. (1M)
14	**If X and Y are two random variables having joint pdf** $f(x,y)=\frac{1}{8}(6-x-y)$**, 0<x<2, 2<y<4. Find (i)** r_{xy} **(ii)P(X<1 /Y<3) (8 M)** BTL5 **Answer : Page : 2.109 – Dr. A. Singaravelu** • $f(x)=\int_2^4 \frac{1}{8}(6-x-y)dy=\frac{6-2x}{4}$ (1M) • $f(y)=\int_0^2 \frac{1}{8}(6-x-y)dy=\frac{10-2y}{8}$ (1M) • $E(X)=\int x\,f(x)dx=\frac{5}{6}$ (1M) • $E(Y)=\int y\,f(y)dy=\frac{17}{6}$ (1M) • $E(X^2)=\int x^2 f(x)dx=1$ (1M) • $E(Y^2)=\int y^2 f(y)dy=\frac{25}{3}$ (1M)

	• $E(XY)=\int\int x\, f(x)dx=\frac{7}{3}$ (1M) • $\sigma_x^2=\frac{11}{36},\ \sigma_y^2=\frac{11}{36}$ (1M) • $r_{xy}=\frac{Cov(X,Y)}{\sigma_x.\sigma_y}=-\frac{1}{11}$ (1M)
15	**The two lines of regression are *8x – 10y +66 = 0; 40x -18y -214 = 0.* The variance of ‘x’ is 9. Find th4e mean values of ‘x’ and ‘y’. Also find the correlation coefficient between ‘x’ and ‘y’.(8 M) (Apr/May 2015, May/June 2016)** BTL4 **Answer: Page : 2.129 – Dr.A. Singaravelu** • $\bar{x}=13$, $\bar{y}=17$ • From first equation $x=\frac{10}{8}y-\frac{66}{8} \Rightarrow b_{xy}=\frac{10}{8}$. (2M) • From the second equation $y=\frac{40}{18}x-\frac{214}{18} \Rightarrow b_{yx}=\frac{40}{18}$. (1M) • Correlation coefficient $r=1.66$ which is not less than 1. (1M) • Now, From first equation $y=\frac{8}{10}x+\frac{66}{10} \Rightarrow b_{yx}=\frac{8}{10}$. (1M) • From the second equation $x=\frac{18}{40}y-+\frac{214}{40} \Rightarrow b_{yx}=\frac{18}{40}$. (1M) • Correlation coefficient $r=\pm 0.6$. (2M)
16	**If the pdf of a two dimensional random variable (X,Y) is given by $f(x,y)=x+y,;0\le(x,y)\le 1$. Find the pdf of U=XY.(8 M) (Apr/May 2015, Nov/Dec 2017)** BTL4 **Answer : Page : 2.156 – Dr.A.Singaravelu** • Take $u=xy$and $v=y$. • $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{v}$. (2M) • $f(u,v)=\lvert J\rvert\, f(x,y)=1+\frac{u}{v^2}$. (3M) • $f(u)=\int_u^1\left(1+\frac{u}{v^2}\right)dv=2-2u$. (3M)
17	**Let (X,Y) be a two-dimensional non-negative continuous random variable having the joint density $f(x,y)=\begin{cases}4xye^{-\left(x^2+y^2\right)} & ,\ x,y\ge 0\\ 0 & ,elsewhere\end{cases}$. Find the density function of $U=\sqrt{X^2+Y^2}$. (8 M) (May/June 2016, Apr/May 2018)** BTL5 **Answer : Page : 2.179 – Dr.A. Singaravelu** • Take $u^2=x^2+y^2$, $v=x$

	• $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{u}{\sqrt{u^2-v^2}}$. (2M) • $f(u,v)=\lvert J\rvert\, f(x,y)=4uve^{-u^2}$. (3M) • $f(u)=\int_0^u\left(4uve^{-u^2}\right)dv=2u^3e^{-u^2}$. (3M)
18	**If X and Y are independent random variables with pdf** $e^{-x}, x\geq 0$; $e^{-y}, y\geq 0$ **respectively. Find the density function of** $U=\frac{X}{X+Y}$ **and** $V=X+Y$**. Are X and Y independent? (8 M) (Nov/Dec 2013, Apr/May 2017, Nov/Dec 2017)** BTL5 **Answer : Page : 2.176- Dr. A. Singaravelu** • Take $U=\frac{X}{X+Y}$ and $V=X+Y$. • $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=v$. (2M) • $f(u,v)=\lvert J\rvert\, f(x,y)=ve^{-v}$. (1M) • $f(u)=\int_0^\infty\left(ve^{-v}\right)dv=1$ (2M) • $f(v)=\int_0^\infty\left(ve^{-v}\right)du=ve^{-v}$. (2M) • $f(u).f(v)=1.\,ve^{-v}=ve^{-v}=f(u,v)$. (1M)
19	**If X_1, X_2, …, X_n are Poisson variables with parameter** $\lambda=2$**, use the central limit theorem to estimate P(120 < S_n< 160) where S_n = X_1+ X_2+ …+X_n and n=75. (8M)** BTL5 **Answer:Page: 2.187-Dr.A. Singaravelu** • $n\mu=150; n\sigma=\sqrt{150}$. (1M) • $z=\frac{S_n-n\mu}{\sigma\sqrt{n}}$; If $S_n=120, z=\frac{-30}{\sqrt{150}}$. (2M) • If $S_n=160, z=\frac{10}{\sqrt{150}}$. (2M) • $P(120<S_n<160)=P(-2.45\leq S_n\leq 0.85)=P(-2.45\leq S_n\leq 0)+P(0\leq S_n\leq 0.85)=0.7866$. (3M)

	UNIT III – Random Proccesses

	Classification – Stationary process – Markov process - Markov chain - Poisson process – Random telegraph process.
	PART *A
Q.No	**Questions**
1.	**Define a random process and give an example. (May/June 2016)** BTL1 A random process is a collection of random variables {X(s,t)} that are functions of a real variable, namely time 't' where $s \in S$ (Sample space) and $t \in T$ (Parameter set or index set). Example: $X(t) = A\cos(\omega t + \theta)$ where θ isuniformly distributed in $(0, 2\pi)$, where 'A' and 'ω' are constants.
2	**State the two types of stochastic processes.**BTL1 The four types of stochastic processes are Discrete random sequence, Continuous random sequence, Discrete random process and Continuous random process.
3	**Define Stationary process with an example.(May/June 2016)** BTL1 If certain probability distribution or averages do not depend on 't', then the random process {X(t)} is called stationary process. Example: A Bernoulli process is a stationary process as the joint probability distribution is independent of time.
4	**Define first Stationary process.(Nov/Dec 2015)** BTL1 A random process {X(t)} is said to be a first order stationary process if $E[X(t)] = \mu$ is a constant.
5	**Define strict sense and wide sense stationary process.(Nov/Dec 2015, Apr/May 2017, Nov/Dec 2017)** BTL1 A random process is called a strict sense stationary process or strongly stationary process if all its finite dimensional distributions are invariant under translation of time parameter. A random process is called wide sense stationary or covariance stationary process if its mean is a constant and auto correlation depends only on the time difference.
6	**In the fair coin experiment we define {X(t)} as follows** $X(t) = \begin{cases} \sin \pi t & ,if\ head\ shows \\ 2t & ,if\ tail\ shows \end{cases}$ **.Find E[X(t)] and find F(x,t) for t = 0.25. (Nov/Dec 2016)** BTL3 $P[X(t) = \sin \pi t] = \frac{1}{2}$, $P[X(t) = 2t] = \frac{1}{2}$ $E[X(t)] = \sum X(t) P[X(t)] = \sin \pi t \left(\frac{1}{2}\right) + 2t\left(\frac{1}{2}\right) = \frac{1}{2}\sin \pi t + t$ $When\ t = 0.25$, $P[X(0.25) = \sin \pi(0.25)] = P\left[X(0.25) = \frac{1}{\sqrt{2}}\right] = \frac{1}{2}$ $P[X(t) = 2(0.25)] = P\left[X(t) = \frac{1}{2}\right] = \frac{1}{2}$ Hence F(x,t) for t= 0.25 is given by $F(x,t) = \begin{cases} 0 & , x < 0 \\ \frac{1}{2} & , \frac{1}{2} \le x < \frac{1}{\sqrt{2}} \\ 1 & , x \ge \frac{1}{\sqrt{2}} \end{cases}$

7	**Prove that a first order stationary random process has a constant mean. (Apr/May 2011)** BTL3 f[X(t)] = f[X(t+h)] as the process is stationary. $E[X(t)]=\int X(t)\, f[X(t+h)]d(t+h)$ $t+h=u \Rightarrow d(t+h)=du$ $Put \quad =\int X(u)\, f[X(u)]du$ $=E[X(u)]$ Therefore, E[X(t+h)] = E[X(t)] Therefore, E[X(t)] is independent of ‘t’. Therefore , E[X(t)] is a constant.
8	**What is a Markov process. Give an example.(Nov/Dec 2014, Apr/May 2015, May/June 2016,Apr/May 2018)** BTL1 Markov process is one in which the future value is independent of the past values, given the present value. (i.e.,)A random process X(t) is said to be a Markov process if for every $t_0< t_1< t_2 <\ldots t_n$, $P\{X(t_n)\le x_n / X(t_{n-1})=x_{n-1}, X(t_{n-2})=x_{n-2}, ..., X(t_0)=x_0\} \Rightarrow P\{X(t_n)\le x_n / X(t_{n-1})=x_{n-1}\}$. Example: Poisson process is a Markov process. Therefore, number of arrivals in (0,t) is a Poisson process and hence a Markov process.
9	**Define Markov chain. When it is called homogeneous? Also define one-step transition probability. (Apr/May 2010)** BTL1 • If $\forall n,\ P[X_n=a_n / X_{n-1}=a_{n-1}, X_{n-2}=a_{n-2}, ..., X_0=a_0]=P[X_n=a_n / X_{n-1}=a_{n-1}]$ then the process $\{X_n\}$ n = 0,1,2,… is called a Markov chain. • In a Markov chain if the one-step transition probability $P[X_n=a_n / X_{n-1}=a_{n-1}]=P_{ij}(n-1,n)$ independent of the step ‘n’. (i.e.,) $P_{ij}(n-1,n)=P_{ij}(m-1,m)$ for all m,n and I,j. Then the Markov chain is said to be homogeneous. • The conditional probability $P[X_n=a_j / X_{n-1}=a_j]$ is called the one step transition probability from state a_i to state a_j at the nth step.
10	**Define Poisson process.(Nov/Dec 2017)** BTL1 If X(t) represents the number of occurrences of a certain event in (0,t), then the discrete process {X(t)} is called the Poisson process provided the postulates are satisfied: $P[1\ occurrence\ in\ (t,t+\Delta t)]=\lambda\Delta t+0(\Delta t)$ $P[0\ occurrence\ in\ (t,t+\Delta t)]=1-\lambda\Delta t+0(\Delta t)$ $P[2\ occurrence\ in\ (t,t+\Delta t)]=0(\Delta t)$ X(t) is independent of the number of occurrences of the event in any interval prior and after the interval (0,t) The provability that the event occurs a specified number of times in (t_0, t_0+t) depends only on ‘t’, but not on ‘t_0’.
11	**State any two properties of Poisson process. (Nov/Dec 2015, Apr/May 2018)** BTL1 • The Poisson process is a Markov process • Sum of two different Poisson process is a Poisson process • Difference of two different Poisson process is not a Poisson process
12	**If the customers arrive at a bank according to a Poisson process with mean rate 2 per minute, find the probability that during a 1-minute interval no customers arrive. (Apr/May 2017)** BTL3 Mean arrival rate = λ = 2

	The probability of Poisson process is $P[X(t)=n]=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $P[X(t)=0]=\frac{e^{-2}(2)^0}{0!}=e^{-2}=0.1353$.
13	**Prove that the sum of two independent Poisson process is a Poisson process.(Nov/Dec 2012, Apr/May 2015, Apr/May 2017)** BTL5 $Let\ X(t)=\left[X_1(t)+X_2(t)\right]$ $E[X(t)]=E\left[X_1(t)+X_2(t)\right]=E\left[X_1(t)\right]+E\left[X_2(t)\right]$ $=\lambda_1 t+\lambda_2 t=(\lambda_1+\lambda_2)t$ $E[X^2(t)]=E\left[X_1(t)+X_2(t)\right]^2=E\left[X_1^2(t)+2X_1(t)X_2(t)+X_2^2(t)\right]$ $=E\left[X_1^2(t)\right]+2E\left[X_1(t)\right]E\left[X_2(t)\right]+E\left[X_2^2(t)\right]$ $=\lambda_1^2t^2+\lambda_1 t+2(\lambda_1 t)(\lambda_2 t)+\lambda_2^2t^2+\lambda_2 t$ $=(\lambda_1+\lambda_2)^2t^2+(\lambda_1+\lambda_2)t$ Therefore $X(t)=\left[X_1(t)+X_2(t)\right]$is a Poisson process.
14	**Prove that the sum of two independent Poisson process is a Poisson process.** BTL5 $Let\ X(t)=\left[X_1(t)-X_2(t)\right]$ $E[X(t)]=E\left[X_1(t)+-X_2(t)\right]=E\left[X_1(t)\right]-E\left[X_2(t)\right]$ $=\lambda_1 t-\lambda_2 t=(\lambda_1-\lambda_2)t$ $E[X^2(t)]=E\left[X_1(t)-X_2(t)\right]^2=E\left[X_1^2(t)-2X_1(t)X_2(t)+X_2^2(t)\right]$ $=E\left[X_1^2(t)\right]-2E\left[X_1(t)\right]E\left[X_2(t)\right]+E\left[X_2^2(t)\right]$ $=\lambda_1^2t^2+\lambda_1 t-2(\lambda_1 t)(\lambda_2 t)+\lambda_2^2t^2+\lambda_2 t$ $=(\lambda_1-\lambda_2)^2t^2+(\lambda_1+\lambda_2)t$ $\neq(\lambda_1-\lambda_2)^2t^2+(\lambda_1-\lambda_2)t$ Therefore $X(t)=\left[X_1(t)-X_2(t)\right]$is nota Poisson process.
15	**Patients arrive randomly and independently at a doctor's consulting room form 8 A.M at an average rate of 1 for every 5 minutes. The waiting room can hold 12 persons. What is the probability that the room will be full when the doctor arrives at 9 A.M?. (Nov/Dec 2016)** BTL3 Given $\lambda=\frac{1}{5}\ per\ \min=\frac{1}{5}\times 60=12\ per\ hour$ The probability law of Poisson process is $P[X(t)=n]=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $P[X(1)=12]=\frac{e^{-12}(12)^{12}}{12!}=0.1144$
16	**Define Semi- Random telegram signal process. (Apr/May 2015)** BTL1 If N(t) represents the number of occurrences of a specified event in (0,t) and X(t) = $(-1)^{N(t)}$, then {X(t)} is called a semi-random telegraph signal process.

17

Define Random telegraph process. BTL1

A random telegraph process is a discrete random process X(t) satisfying the following conditions:

- X(t) assumes only one of the two possible values 1 or -1 at any time 't', randomly
- X(0) = 1 or -1 with equal probability ½.
- The number of level transitions or flips, $N(\tau)$, from one value to another occurring in any interval of length τ is a Poisson process with rate λ so that the probability of exactly 'r' transitions is $P[N(\tau)=r]=\dfrac{e^{-\lambda\tau}(\lambda\tau)^r}{r!}$, $r=0,1,2,...$

18

Write the properties of Random telegraph process. BTL1

- P[X(t)=1] = $\dfrac{1}{2}$ = P[X(t)= - 1] for any t >0
- E[X(t)] = 0 and Var[X(t)] = 1
- X(t) is a WSS process

19

Consider the random process $X(t)=\cos(t+\phi)$ where ϕ is a random variable with density function $f(\phi)=\dfrac{1}{\pi},\ -\dfrac{\pi}{2}<\phi<\dfrac{\pi}{2}$. Check whether or not the process is stationary. BTL3

$$E[X(t)]=\int_{-\infty}^{\infty} X(t)\, f(\phi)\, d\phi$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi)\, \frac{1}{\pi}\, d\phi$$

$$=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t+\phi)\, d\phi$$

$$=\frac{1}{\pi}\left[\sin(t+\phi)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

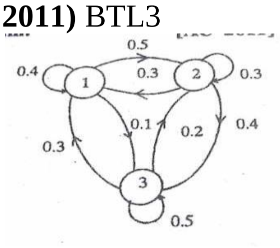
$$=\frac{1}{\pi}\left[\sin(\frac{\pi}{2}+t)-\sin(-\frac{\pi}{2}+t)\right]$$

$$=\frac{1}{\pi}\left[\cos(t)+\cos(t)\right]=\frac{2}{\pi}\cos(t)$$

Therefore E[X(t)] is not a constant. Hence X(t) is not stationary.

20

Find the transition probability matrix of the process represented by the transition diagram. (Apr/May 2011) BTL3



<table>
<tr><td></td><td>

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

</td></tr>
<tr><td>21</td><td>

If the tpm of the markov chain is $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$**, find the steady-state distribution of the chain.** BTL5

Given : P= $\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Let the steady- state probability distribution be $\pi = (\pi_1 \quad \pi_2)$ we have

$\pi P = \pi$ ....(1)

$\pi_1 + \pi_2 = 1$...........(2)

$(1) \Rightarrow (\pi_1 \quad \pi_2)\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = (\pi_1 \quad \pi_2)$

$\left[\pi_1(0) + \pi_2\left(\frac{1}{2}\right) \quad \pi_1(1) + \pi_2\left(\frac{1}{2}\right)\right] = (\pi_1 \quad \pi_2)$

$\Rightarrow \left[\pi_2\left(\frac{1}{2}\right) \quad \pi_1 + \pi_2\left(\frac{1}{2}\right)\right] = (\pi_1 \quad \pi_2)$

$\Rightarrow \frac{1}{2}\pi_2 = \pi_1$............(3)

$\pi_1 + \pi_2\left(\frac{1}{2}\right) = \pi_2$..........(4)

Now (2) $\Rightarrow \pi_1 + \pi_2 = 1$, *substitute* (3) *in* (2)

$\Rightarrow \frac{1}{2}\pi_2 + \pi_2 = 1 \Rightarrow \frac{3}{2}\pi_2 = 1 \Rightarrow \pi_2 = \frac{2}{3}$

Sub π_2 *in* (3), $\frac{1}{2}.\frac{2}{3} = \pi_1 \Rightarrow \pi_1 = \frac{1}{3}$

The steady state distribution of the chain is $\pi = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

</td></tr>
<tr><td>22</td><td>

Let $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ **be a stochastic matrix. Check if it is regular. (Nov/Dec 2016)** BTL4

$A^2 = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

Since all the entries of A^2 are positive , 'A' is regular.

</td></tr>
<tr><td>23</td><td>

What is the autocorrelation function of the Poisson process. Is Poisson process stationary? BTL2

Let X(t) be a Poisson process then $P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ n=0,1,2,…

</td></tr>
</table>

	Autocorrelation function $R_{xx}(t_1,t_2)=\lambda^2 t_1 t_2+\lambda \min\{t_1,t_2\}$ Since $R_{xx}(t_1,t_2)$ is not a function of time difference t_1-t_2, Poisson process is not stationary.
24	**When is a Random process said to be evolutionary. Give an example. (Apr/May 2015)** (BTL1) A random process that is not stationary at any sense is called evolutionary process. Semi-random telegraph signal process is an example of evolutionary random process.
25	**Define irreducible Markov chain and state Chapman-Kolmogorov theorem.** BTL1 A Markov chain is said to be irreducible if every state can be reached from every other state, where $p_{ij}^{(n)}>0$ for some 'n' and for all 'i' and 'j'. If 'P' is the tpm of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n. (i.e.,) $\left[P_{ij}^{(n)}\right]=\left[P_{ij}\right]^n$.
	Part*B
1	**The process {X(t)} whose probability distribution under certain conditions is given by,** $P\{X(t)=n\}=\dfrac{(at)^{n-1}}{(1+at)^{n+1}},\ n=1,2...$ $=\dfrac{at}{1+at},\ n=0$ **Show that it is not stationary(evolutionary). (8M)(Nov/Dec 2014, Nov/Dec 2016, Apr/May 2018)** BTL5 **Answer: Page: 3.33 –Dr. A. Singaravelu** • $E[X(t)]=\sum_{n=0}^{\infty} n\,p_n=0+(1)\dfrac{1}{(1+at)^2}+(2)\dfrac{at}{(1+at)^3}+\dots=1.$ (3M) • $E[X^2(t)]=\sum_{n=0}^{\infty} n^2 P_n=\sum_{n=0}^{\infty}([n(n+1)-n]P_n=1+2at$. (3M) • $Var[X(t)]=E[X^2(t)]-E[X(t)]=2at\neq cons\tan t$. (2M)
2	**If the random process X(t) takes the value -1 with probability $\frac{1}{3}$ and takes the value 1 with probability $\frac{2}{3}$, find whether X(t) is a stationary process or not. (6M)(Apr/May 2017)** BTL4 **Answer:Page: 3.12 – Dr. G. Balaji** <table><tr><td>X(t)=n</td><td>-1</td><td>1</td></tr><tr><td>P_n</td><td>1/3</td><td>2/3</td></tr></table> • $E[X(t)]=\sum_{n=-1}^{1} n\,P_n=\frac{1}{3}$ (2M) • $E[X^2(t)]=\sum_{n=-1}^{1} n^2 P_n=1$ (2M) • $Var[X(t)]=E[X^2(t)]-E[X(t)]=\frac{8}{9}$ = constant. (2M)
3	**Show that the process $X(t)=A\cos(\omega t+\theta)$ where A, ω are constants, θ is uniformly distributed in $(-\pi,\pi)$ is wide sense stationary. (8M) (May/June 2016, Nov/Dec 2016)** BTL5 **Answer:Page: 3.15-Dr. A. Singaravelu**

	• $E[X(t)]=\int_{-\infty}^{\infty} X(t)\, f(\theta)\, d\theta=\int_{-\pi}^{\pi} A\cos(\omega t+\theta)\frac{1}{2\pi}\, d\theta=0=cons\tan t$ (2M) • $R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]=E[A\cos(\omega t+\theta).A\cos(\omega(t+\tau)+\theta)]$ (1M) • $E[A\cos(\omega t+\theta).A\cos(\omega(t+\tau)+\theta)]=\frac{A^2}{2}\{E(\cos\omega\tau)+E[\cos(2\omega t+2\theta+\omega\tau)]\}$ (2M) • $E[\cos(2\omega t+2\theta+\omega\tau)]=0$ (2M) • $R_{XX}(t,t+\tau)=\frac{A^2}{2}\cos\omega\tau$ =a function of τ. (1M)
4	**Show that the process $X(t)=A\cos(\omega t+\theta)$ where A, ω are constants, θ is uniformly distributed in $(0,2\pi)$ is WSS. (8M) (Nov/Dec 2017)** BTL5 **Answer:Page: 3.24-Dr. G. Balaji** • $E[X(t)]=\int_{-\infty}^{\infty} X(t)\, f(\theta)\, d\theta=\int_{0}^{2\pi} A\cos(\omega t+\theta)\frac{1}{2\pi}\, d\theta=0=cons\tan t$ (2M) • $R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]=E[A\cos(\omega t+\theta).A\cos(\omega(t+\tau)+\theta)]$ (1M) • $E[A\cos(\omega t+\theta).A\cos(\omega(t+\tau)+\theta)]=\frac{A^2}{2}\{E(\cos\omega\tau)+E[\cos(2\omega t+2\theta+\omega\tau)]\}$ (2M) • $E[\cos(2\omega t+2\theta+\omega\tau)]=0$ (2M) • $R_{XX}(t,t+\tau)=\frac{A^2}{2}\cos\omega\tau$ =a function of τ. (1M)
5	**Show that the process $X(t)=A\cos\lambda t+B\sin\lambda t$ is strict sense stationary of order 2. A and B are random variables if E[A] = E[B] = 0 ; E[A^2] = E[B^2]; E[AB] = 0.** **(OR)** **If $X(t)=A\cos\lambda t+B\sin\lambda t$, $t\geq 0$ is a random process where A and B are independent $N(0,\sigma^2)$ random variables. Examine the WSS process of X(t). (8M) (Apr/May 2015, Apr/May 2017)** BTL5 **Answer:Page: 3.13-Dr. A. Singaravelu** • $E\{X(t)\}=E\{A\cos\lambda t+B\sin\lambda t\}=0=cons\tan t$ (2M) • $R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]=E\{[A\cos\lambda t+B\sin\lambda t][A\cos\lambda(t+\tau)+B\sin\lambda(t+\tau)]\}$ (2M) • $R_{XX}(t,t+\tau)=K^2[\cos\lambda t\cos\lambda(t+\tau)+\sin\lambda t\sin\lambda(t+\tau)]=K^2\cos\lambda\tau$ (4M)
6	**A random variable {X(t)} is defined by $X(t)=A\cos t+B\sin t, -\infty<t<\infty$ where A and B are independent random variables each of which has a value -2 with probability $\frac{1}{3}$ and a value 1 with probability $\frac{2}{3}$. Show that X(t) is wide sense stationary. (8M) (Nov/Dec 2015, Apr/May 2017, Apr/May 2018)** BTL5 **Answer:Page: 3.44-Dr. G. Balaji** • $E[A]=\sum A_i\, P(A_i)=0$ (1M) • $E[B]=\sum B_i\, P(B_i)=0$ (1M) • $E[A^2]=\sum A_i^2\, P(A_i)=2$ (1M) • $E[B^2]=\sum B_i^2\, P(B_i)=2$ (1M) • $E[X(t)]=E[Y\cos t+Z\sin t]=0=cons\tan t$ (2M) • $R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]=E[(Y\cos t_1+Z\sin t_1)(Y\cos t_2+Z\sin t_2)]=2\cos\tau$ (2M)

7	**The transition probability matrix of a Markov chain {X$_n$}, n=1,2,… having 3 states 1,2 and 3 is** $P=\begin{bmatrix}0.1 & 0.5 & 0.4\\0.6 & 0.2 & 0.2\\0.3 & 0.4 & 0.3\end{bmatrix}$ **and the initial distribution is** $P^{(0)}=(0.7 \quad 0.2 \quad 0.1)$**. Find (i)** $P\{X_2=3\}$ **and (ii)** $P\{X_3=2, X_2=3, X_1=3, X_0=2\}$**.** **Answer:Page: 3.60-Dr. A. Singaravelu** • $P^{(1)}=P^{(0)}P=[0.7 \quad 0.2 \quad 0.1]\begin{bmatrix}0.1 & 0.5 & 0.4\\0.6 & 0.2 & 0.2\\0.3 & 0.4 & 0.3\end{bmatrix}=[0.22 \quad 0.43 \quad 0.35]$ (2M) • $P^{(2)}=P^{(1)}P=[0.22 \quad 0.43 \quad 0.35]\begin{bmatrix}0.1 & 0.5 & 0.4\\0.6 & 0.2 & 0.2\\0.3 & 0.4 & 0.3\end{bmatrix}=[0.385 \quad 0.336 \quad 0.279]$ (2M) • $P\{X_2=3\}=0.279$ (1M) • $P\{X_3=2, X_2=3, X_1=3, X_0=2\}=P_{32}^1\, P_{33}^1\, P_{23}^1\, P[X_0=2]=0.0048$ (3M)
8	**A man either drives a car or catches a train to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drive to work if and only if a 6 appeared. Find (i) The probability that he drives to work in the long run and (ii) The probability that he takes a train on the third day. (8M) (May/June 2016, Nov/Dec 2017)** BTL4 **Answer:Page: 3.71-Dr. A. Singaravelu** • $P=\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ (2M) • $\pi=(\pi_1 \quad \pi_2)=\begin{pmatrix}\frac{1}{3} & \frac{2}{3}\end{pmatrix}$ (3M) • $P^{(2)}=P^{(1)}P=\begin{pmatrix}\frac{1}{12} & \frac{11}{12}\end{pmatrix}$ (1M) • $P^{(3)}=P^{(2)}P=\begin{pmatrix}\frac{11}{24} & \frac{13}{22}\end{pmatrix}$ (2M)
9	**If {X$_n$; n=1,2,3…} be a Markov chain on the space S={1,2,3} with one-step** $\begin{bmatrix}0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\\ 1 & 0 & 0\end{bmatrix}$**.Sketch the transition diagram. Is the chain irreducible? Explain. Is the chain ergodic? Explain. (8M) (May/June 2013, Nov/Dec 2014)** BTL4 **Answer:Page:3.141-Dr. G. Balaji** • $P^4=P^3P=P.P=P^2$ (1M) • $P^5=P^4P=P^2.P=P^3=P$ (1M) • 1st state $P_{00}^{(2)}>0, P_{00}^{(4)}>0, P_{00}^{(6)}>0$ ....$\Rightarrow d_i=GCD(2,4,6,...)=2$ (1M) • 2nd state $P_{11}^{(2)}>0, P_{11}^{(4)}>0, P_{11}^{(6)}>0$ ....$\Rightarrow d_i=GCD(2,4,6,...)=2$ (1M)

	• 3^{rd} state $P_{22}^{(2)} > 0, P_{22}^{(4)} > 0, P_{22}^{(6)} > 0 \; .... \Rightarrow d_i = GCD(2,4,6,...) = 2$ (1M) • The states are aperiodic with period 2. • We find $P_{ij}^{(n)} > 0$. So the Markov chain is irreducible (2M) • The chain is finite and irreducible so it is non- null persistant. But not ergodic. (1M)
10	**Find the mean, variance and auto correlation of Poisson process. (8M) (May/June 2014, Apr/May 2015)** BTL2 **Answer: Page:3.93- Dr. A. Singaravelu** • The probability of Poisson distribution is $P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, n=0,1,2,… (1M) • $E[X(t)] = \sum_{x=0}^{\infty} x \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \lambda t$ (2M) • $E[X^2(t)] = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda t}(\lambda t)^n}{n!} = (\lambda t)^2 + \lambda t$ (2M) • $Var[X(t)] = \lambda t$ (1M) • $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$ (2M)
11	**(i) Prove that the interval between two successive occurrences of a Poisson process with parameter λ has an exponential distribution.** **(ii)Show that Poisson process is a Markov process. (8M) (Apr/May 2018)**BTL5 **Answer: Page:3.98- Dr. A. Singaravelu** (i) • P(T > t) = P(E_{i+1} did not occur in (t_i, t_{i+1})= P(X(t)=0) = $e^{-\lambda t}$ (1M) • $F(t) = P(T \le t) = 1 - P(T > t) = 1 - e^{-\lambda t}$ (2M) • The pdf of T is given by $\lambda e^{-\lambda t}$ which is an exponential distribution. (1M) (ii) • $P[X(t_3) = n_3 / X(t_2) = n_2 ; X(t_1) = n_1] = \frac{e^{-\lambda(t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$ (3M) • $P[X(t_3) = n_3 / X(t_2) = n_2 ; X(t_1) = n_1] = P[X(t_3) = n_3 / X(t_2) = n_2]$ which is Markov process. (1M)
12	**Suppose that customers arrive at a bank according to a Poisson process with mean rate of 3 per minute; find the probability that during a time interval of 2 min (i) exactly 4 customers arrive and (ii) more than 4 customers arrive. (iii) fewer than 4 customers arrive. (8M) (Nov/Dec 2015)** BTL5 **Answer: Page:3.100- Dr. A. Singaravelu** • The probability of Poisson distribution is $P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, n=0,1,2,… (1M) • P[4 customers arrive in 2 min time interval] = P{X(2)=4} = 0.1339 (2M) • P[More than 4 customers arrive in 2 min interval] =P{X(2)>4} = $1 - P[X(2) \le 4]$= 0.715(3M) • P[Fewer than 4 customers arrive in 2 min interval] = P{X(2)<4} = 0.1512. (2M)
13	**A fisherman catches a fish at a Poisson rate of 2 per hour from a large lake with lots of fish. If he starts fishing at 10.00 a.m. What is the probability that he catches one fish by 10.30 a.m and three fishes by noon? (8M) (Apr/May 2017)**BTL5 **Answer: Classwork**

	• The probability of Poisson distribution is $P\{X(t)=n\}=\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, n=0,1,2,… (2M) • P[He catches one fish by 10.30 a.m] =P[X(0.5)=1] $=\dfrac{e^{-1}(1)^1}{1!}$ =0.3679 (3M) • P[He catches three fishes by noon] = P[X(2) = 3] = $\dfrac{e^{-4}(4)^3}{3!}=0.1954$ (2M)
14	**A hard disk fails in a computer system and it follows Poisson process with mean rate of 1 per week. Find the probability that 2 weeks have elapsed since the last failure. If there are 5 extra hard disks and the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks. (8M) (Nov/Dec 2017)** BTL5 **Answer: Page:3.102- Dr. A. Singaravelu** • The probability of Poisson distribution is $P\{X(t)=n\}=\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, n=0,1,2,… (2M) • P[No failure in 2 weeks since last failure] = P[X(2)=0] = e^{-2} = 0.135 (3M) • $P[X(10)\le 5]=P[X(10)=0]+[X(10)=1]+[X(10)=2]+[X(10)=3]+[X(10)=4]+[X(10)=5]=0.067$ (3M)
15	**If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is (i) more than 1 minute, (ii) between 1 min and 2 min and (iii) 4 min or less. (8M) (May/June 2012)** BTL5 **Answer: Page: 3.100- Dr. A. Singaravelu** • Using inter arrival property of Poisson process, $f(t)=\lambda e^{-\lambda t}$ (1M) • $P(T>1)=\int_1^{\infty}2e^{-2t}dt=0.135$ (2M) • $P(1<T<2)=\int_1^{2}2e^{-2t}dt=0.117$ (2M) • $P(T\le 4)=\int_0^{4}2e^{-2t}dt=1$ (3M)
16	**If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent Poisson process with parameter λ_1 and λ_2 respectively, show that $P[X_1(t)=x / X_1(t)+X_2(t)=n]$ is Binomial where $P=\dfrac{\lambda_1}{\lambda_1+\lambda_2}$.(8M) (Apr/May 2018)** BTL5 **Anwer: Page: 3.84-Dr G. Balaji** • $P[X_1(t)=x/X_1(t)+X_2(t)=n]=\dfrac{P[\{X_1(t)=x\}\cap\{X_1(t)+X_2(t)=n\}]}{P(X_1(t)+X_2(t)=n)}$ (3M) • $P[X_1(t)=x/X_1(t)+X_2(t)=n]=\dfrac{\dfrac{e^{-\lambda_1 t}(\lambda_1 t)^x}{x!}\cdot\dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-x}}{(n-x)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^n}{n!}}$ (3M) • $P[X_1(t)=x/X_1(t)+X_2(t)=n]=nC_xP^xq^{n-x}$ where $P=\dfrac{\lambda_1}{\lambda_1+\lambda_2}$ and $q=\dfrac{\lambda_2}{\lambda_1+\lambda_2}$ (2M)
17	**Define semi-random telegraph signal process and random telegraph signal process and prove that the former is evolutionary and the latter is wide sense stationary(Covariance stationary process). (16M)**

(Nov/Dec 2013, Nov/Dec 2017, Apr/May 2015, Apr/May 2017) BTL5

Answer: 3.106- -Dr.A. Singaravelu

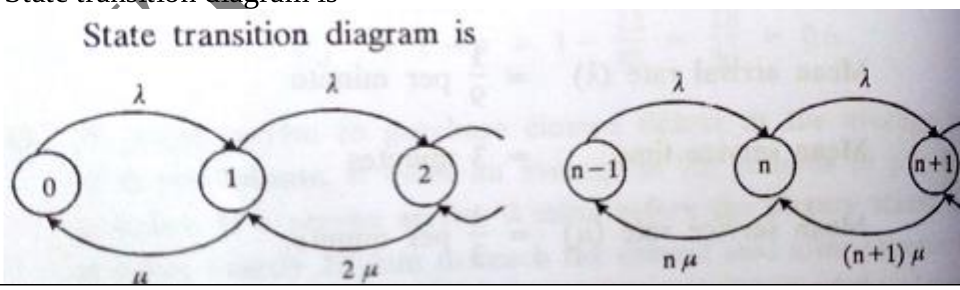
- A random telegraph process is a discrete random process X(t) satisfying the following conditions:
  X(t) assumes only one of the two possible values 1 or -1 at any time 't', randomly
  X(0) = 1 or -1 with equal probability ½.

The number of level transitions or flips, $N(\tau)$, from one value to another occurring in any interval of length τ is a Poisson process with rate λ so that the probability of exactly 'r' transitions is $P[N(\tau)=r]=\frac{e^{-\lambda\tau}(\lambda\tau)^r}{r!}$, $r=0,1,2,...$

(2M)

- If N(t) represents the number of occurrences of a specified event in (0,t) and $X(t) = (-1)^{N(t)}$, then {X(t)} is called a semi-random telegraph signal process. (2M)
- $P\{X(t)=1\} = P\{N(t) \text{ is even}\} = e^{-\lambda t}\cosh\lambda t$ (1M)
- $P\{X(t)=-1\} = P\{N(t) \text{ is odd}\} = e^{-\lambda t}\, Sinh\,\lambda t$ (1M)
- $E[X(t)] = e^{-2\lambda t}$ (1M)
- $P[X(t_1)=1, X(t_2)=1] = P[X(t_1)=1/X(t_2)=1]\times P[X(t_2)=1] = e^{-\lambda\tau}\cosh\lambda\tau\, e^{-\lambda t_2}\cosh\lambda t_2$ (1M)
- $P[X(t_1)=-1, X(t_2)=-1] = e^{-\lambda\tau}\cosh\lambda\tau\, e^{-\lambda t_2}\, Sinh\,\lambda t_2$ (1M)
- $P[X(t_1)=1, X(t_2)=-1] = e^{-\lambda\tau}\sinh\lambda\tau\, e^{-\lambda t_2}\, Sinh\,\lambda t_2$ (1M)
- $P[X(t_1)=-1, X(t_2)=1] = e^{-\lambda\tau}\sinh\lambda\tau\, e^{-\lambda t_2}\cosh\lambda t_2$ (1M)
- $P[X(t_1)\times X(t_2)=1] = e^{-\lambda\tau}\cosh\lambda\tau$ (1M)
- $P[X(t_1)\times X(t_2)=-1] = e^{-\lambda\tau}\sinh\lambda\tau$
- $R(t_1,t_2)=E[X(t_1)X(t_2)]=e^{-2\lambda(t_2-t_1)}$ (1M)
-
  {X(t)} is evolutionary
- For Random telegraph signal process Y(t), $P(\alpha=1)=\frac{1}{2}$, $P(\alpha=-1)=\frac{1}{2}$ (1M)
- $E(\alpha)=0, E(\alpha^2)=1$ (1M)
- $R_{YY}(t_1,t_2)=E[Y(t_1)Y(t_2)]=E[\alpha^2 X(t_1)X(t_2)]=e^{-2\lambda(t_2-t_1)}$ which is WSS. (1M)
-

	UNIT IVQUEUEINGMODELS
	Markovian queues – Birth and death processes – Single and multiple server queuing models – Little's formula - Queues with finite waiting rooms – Queues with impatient customers : Balking andreneging
	PART * A
Q.No.	**Questions**
1.	**In a given M/M/1/∞/FCFS queue $\rho = 0.6$, what is the probability that the queue contains 5 or more customers.** BTL3 The probability that the queue contains 5 or more customers is given by $P(N \ge 5) = \rho^5$ $\Rightarrow (0.6)^5 = 0.0778$
2.	**Discuss the term: (1) Reneging , (2) Jockeying (APR/MAY 2015)**BTL 1 (1) RENEGING: This occurs when a waiting customers leaves the queue due to impatience. (2) JOCKEYING: Customers may Jockey from one waiting line to another. This is most common in a " Supermarket".
3.	**Define Balking.(APR/MAY 2015)** BTL 1 A customers who leaves the queue because the queue is too long and he has no time or has no sufficient waiting space.
4	**What is the probability that a customer has to wait more than 15 minutes to get his service completed in (M/M/1) : (∞/ FIFO) queue system if $\lambda = 6$ per hour and $\mu = 10$ per hour? (NOV/DEC 2003, 2004, APR/MAY 2009, 2011, 2013, 2015)**BTL3 The probability that the waiting time of a customer in the system exceeds $t = e^{(\mu-\lambda)t}$ Given that $\lambda = 6$ per hour $\mu = 10$ per hour $t = 15\,\text{min} = \frac{1}{4} hr$ The requires probability = $e^{-(10-6)\frac{1}{4}=e^{-1}} = 0.3679$
5	**What is the basic characteristics of a queuing system? (MAY/JUNE 2006, 2013)** BTL2 The basic characteristics of the queuing system are 1) Arrival pattern of customers

	2) Service pattern of servers 3) Queue discipline and 4) System capacity.
6	**Write the basic characteristics of a queuing process. (NOV/DEC 2006, 2010)** BTL1 The basic queuing process describes how customers arrive at and proceed through the queuing system. This means that the basic queuing process describes the operation of a queuing system. 1) The calling population 2) The arrival process 3) The queue configuration 4) The queue discipline and 5) The service mechanism.
7	**Define transient state and steady state queuing system.** BTL1 STEADY STATE: If the characteristics of a queuing system are independent of time. TRANSIENT STATE: If the characteristics of a queuing system are dependent of time.
8	**What do the letters in the symbolic representation (a/b/c): (d/e) of a queuing model represent? (NOV/DEC 2011, 2015)** BTL1 Usually a queuing model is specified and represented symbolically in the form (a/b/c):(d/e), where a – the type of distribution of the number of arrivals per unit time; b – the type of distribution of the service time; c – The number of serves d – The capacity of the system, viz., the maximum queue size e – The queue discipline.
9	**Draw the state transition rate diagram for M/M/C queuing model. (MAY/JUNE 2009, 2011, 2015)**BTL1 Self-service model: Here all units are taken into service on arrival and there is no queue $\lambda_n = \lambda$ $\mu_n = n\mu$. $for \quad n =,1,2,...$ State transition diagram is 
10	**Define effective arrival rate with respect at to an (M/M/1):(k/FIFO) queuing model.(APR/MAY 2011)** BTL1 The effective arrival rate is denoted by $\lambda^{'}$ *or* λ_{eff} and defined by $\frac{\lambda^{'}}{\mu} = 1 - P_0 \quad or \quad \lambda^{'} = \mu(1 - P_0)$
11	**Define Morkovian Queuing models**. BTL1

	Queuing models in which both inter-arrival time and service time which are exponentially distributed are called Markovian queuing models.
12	**Explain the term" TRAFFIC INTENSITY".** BTL2 Utilization factor or traffic intensity is the average function of time tat the serves are being utilized while serving customers. $\rho = \frac{Mean \quad arrival \quad rate \quad (\lambda)}{Mean \quad Service \quad rate \quad (\mu)}$
13	**In (M/M/S):(∞/FIFO), $\lambda = 10/hr$, $\mu = 15$/hr, s=2 Calculate P_0.** BTL3 $P_0 = \frac{1}{\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \left[\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]} = \mathbf{\frac{1}{2}}$
14	**Define Little's formula**. BTL1 $W_s = \frac{1}{\mu - \lambda}$ $W_q = W_s - \frac{1}{\mu}$ $L_S = \lambda W_s$ $L_q = \lambda W_q$
15	**For (M/M1) : (∞ / FIFO) models, write the little's formula.** BTL1 $W_s = \frac{1}{\lambda} L_S$ $W_q = \frac{1}{\lambda} L_q$ $L_S = \frac{\rho}{1-\rho}$ $L_q = L_S - \rho$
16	**Write down the Little's formulas that hold good for the infinite capacity Poisson queue models.** BTL1 $W_s = \frac{1}{\mu - \lambda}$ $W_q = W_s - \frac{1}{\mu}$ $L_S = \lambda W_s$ $L_q = \lambda W_q$

17	**Write the relation among L_s, L_q, W_s & W_q.** BTL1 $W_s = \frac{1}{\lambda} L_S$ $W_q = \frac{1}{\lambda} L_q$ $L_S = \frac{\rho}{1-\rho}$ $L_q = L_S - \rho$
18	**In the usual notation of an M/M/1 queuing system, if λ=12 per hour and μ=24 per hour, find the average number of customers in the system. (MAY/JUNE 2007)**BTL3 $\lambda = 12, \mu = 24$ $L_s = \frac{\lambda}{\mu - \lambda} = \frac{12}{24-12} = 1$
19	**Suppose, customers arrive at a Poisson rate of one per every 12 minutes and that the service time is exponential at a rate of one service per 8 minutes. What is (a) The average number of customers in the system. (b) The average time a customer spends in the system.**BTL5 (a) $L_S = \frac{\rho}{1-\rho}$=2 (b) $W_s = \frac{1}{\lambda} L_S$ =24 minute.
20	**If λ, μ are the rates of arrivals and departure in a M/M/1 queue respectively, give the formula for the probability that there are *n* customers in the queue at any time in steady state.**BTL1 $P_n = \left(\frac{\lambda}{\mu}\right)^n \left[1 - \frac{\lambda}{\mu}\right]$
21	**Arrival rate of telephone calls at a telephone booth is according to Poisson distribution with an average time of 9 minutes between two consecutive arrivals. The length of a telephone call is assumed to be exponentially distributed with mean 3 minutes. Determine the probability that a person arriving at the booth will have to wait.**BTL3 Given : Telephone booth – single server Telephone calls – Infinite capacity The given problem is (M/M/1): (∞/FIFO) Mean arrival rate (λ)= 1/9 per minute Mean service rate (μ) = 1/3 per minute $\rho = \frac{\lambda}{\mu}$=0.33
22	**What is the probability that there are no customers in the (M/M/S): (∞/ FIFO) queuing system? (APR/MAY 2011)**BTL1

	$P_0 = \dfrac{1}{\left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \left[\frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]}$
23	**Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains find the probability that the yard is empty.** BTL3 Given : Yard- single server Trains- Finite capacity Hence this problem comes under the model: (M/M/1):(K/FIFO) Mean arrive rate = $\frac{1}{15}$ per minute Mean service rate = $\frac{1}{33}$ per minute k = 4 $\rho = \frac{\lambda}{\mu} = (\frac{33}{15})$ The probability that the yard is empty (P_0) = $\frac{1-\rho}{1-\rho^{k+1}}$ 0.0237
24	**Write down Little's formulas for the averages waiting time in the system and in the queue for an (M/M/s):(k/FIFO) queuing model.**BTL1 Average waiting time in the system and in the queue $W_S = E[W_s] = \frac{1}{\lambda'} E[N]$ $W_q = E[W_q] = \frac{1}{\lambda'} E[N_q]$
25	**If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a tickets. If a person arrives 2 mins before the picture starts and it takes exactly 1.5 min to reach the correct seat after purchasing the ticket. Can he expect to be seated for the start of the picture?**BTL3 Given: $\lambda = 6/\min$ $\mu = 8/\min$ $W_s = \frac{1}{\mu - \lambda} = \frac{1}{2}\min$ $E[total\ \ time\ \ required\ \ to\ \ purchase\ \ the\ \ ticket\ \ and\ \ to\ \ reach\ \ the\ \ seat] = 2\min$
	Part*B
1	**Customers arrive at one-man barber shop according to a poisson process with a mean inter arrival time of 12 min. customers spend an average of 10 min. in the barber's chair.** **a) What is the expected number of customers in the barber shop and in the queue?** **b) Calculate the % of time of arrival; can walk straight into the barber's chair**

without having to wait?

c) **How much time can customer expect to spend in the barber's shop?**

d) **Management will provide another chair and here another barber. When a customer's waiting time in the shop exceeds 1.25h. How much the average rate of arrivals increase to warrant a second barber?**

e) **What is the average time customers spend in the queue?**

f) **What is the probability that the waiting time in the system is greater than 30 min?**

g) **Calculate the % of customers who have to wait prior to getting into the barber's chair?**

h) **What is the probability that more than 3 customers are in the system?(APR/MAY 2011, 2015)(16M)**BTL5

Answer: Page : 3.6 - Dr. G. Balaji

one man barber shop- single server

customers- infinite capacity

The given problem is (M/M/1) : (∞/FIFO) model

Mean arrival rate (λ)= 1/12 per minute
Mean service rate (μ) = 1/10 per minute

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$$

$$P_0 = 1-\rho = 1-\frac{5}{6} = \frac{1}{6}$$

$$L_s = \frac{\rho}{1-\rho} = 5$$

$$L_q = L_s - \rho = 4.17$$

$$W_s = \frac{1}{\lambda} L_s = 60$$

$$W_q = \frac{1}{\lambda} L_q = 50$$

(a)(i) The expected number of customer in the system = L_s = 5 (2M)

(ii)The expected number of customer in the queue = L_q = 4.17 (2M)

(b)P[a customer walk straight into the barber's chair without having to wait]= P_0 =0.1667

(c) Expected time a customer spends in the (barber shop) system= $W_s = 60$ (2M)

(d) Given $W_s > 1.25h \Rightarrow \lambda_R > \frac{13}{150}$. Hence the arrival rate should increase by $\frac{1}{300}$per min. (2M)

(e) Average waiting time per customer in the queue =W_q =50 min. (2M)

(f) P[waiting time in the system > 30 minutes]=P[W>30] = $e^{-(\mu-\lambda)t} = e^{-0.5} = 0.6065$

	(2M) (g) P[a customer has to wait]= $1-P_0 = \rho = \frac{5}{6} \Rightarrow 83.33\%$. (2M) (h) P[more than 3 customer in the system]= P[N>3]= $\rho^4 = 0.4823$. (2M)
2	**If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 min before the picture starts and if it takes exactly 1.5 min to reach the correct seat after purchasing the ticket.** **a) Can he expect to be seated for the start of the picture?** **b) What is the probability that he will be seated for the start of the picture?** **c) How early must he arrive in order to be 99% of sure of being seated for the start of the picture?(NOV/DEC 2010,2014)(16M)**BTL5 **Answer: Page : 4.30 - Dr. G. Balaji** Ticket counter – Single server People – infinite capacity The given problem is (M/M/1) : (∞/FIFO) model Mean arrival rate (λ)= 6 per minute Mean service rate (μ) = $\frac{1}{7.5}$ per second = 8 per minute. $\rho = \frac{\lambda}{\mu} = \frac{6}{8}$ $P_0 = 1-\rho = 1-\frac{6}{8} = \frac{2}{8}$ $L_s = \frac{\rho}{1-\rho} = 3$ $L_q = L_s - \rho = \frac{9}{4}$ (6M) $W_s = \frac{1}{\lambda} L_s = \frac{1}{2}$ min $W_q = \frac{1}{\lambda} L_q = 3/8$ a) E[Total time required to purchase the ticket and to reach the seat]=[$W_s + 1.5 = \frac{1}{2}\min + 1.5$ =2 min (4M) (b) P[he will be seated for the start of the picture]= P[Total time $\leq$2 min]=0.632. (2M) (c) Given: P[W$\leq$t]=0.99 $P[W > t] = 1-0.99 = 0.01$ $t = 2.3\min$ Therefore, P[Ticket purchasing time<2.3] = 0.99 P[Total time to get the ticket and to go the seat < (2.3+1.5)]=0.99

	Hence, the person must arrive atleast 3.8 minutes early, so as to be 99% sure of seeing the start of the picture. (4M)
3	**A duplicating machine maintained for office use is operated by an office assistant who earns Rs. 5 per hour. The time to complete each job varies according to an exponential distribution with mean 6 min. Assume a poisson input with an average arrival rate of 5 jobs per hour. If an 8 hour day is used as a base, determine** **a) The % idle time of the machine.** **b) The average time a job is in the system and** **c) The average earning per day of the assistant. (NOV/DEC 2008)(16M)**BTL5 **Answer: Page : 4.35- Dr. G. Balaji** Duplicating machine- single server. Job varies- infinite capacity The given problem is (M/M/1) : (∞/FIFO) model Mean arrival rate (λ)= 5 per hour Mean service rate (μ) = $\frac{1}{6}$per minute = 10 per hour. $\rho = \frac{\lambda}{\mu} = \frac{5}{10} = \frac{1}{2}$ $P_0 = 1-\rho = 1-\frac{1}{2} = \frac{1}{2}$ $L_s = \frac{\rho}{1-\rho} = 1$ $L_q = L_s - \rho = \frac{1}{2}$ (6M) $W_s = \frac{1}{\lambda} L_s = \frac{1}{5}\, hour$ $W_q = \frac{1}{\lambda} L_q = \frac{1}{10}$ a) P[the machine is idle] = $\frac{1}{2}$ (3M) b) Average time a job in the system $= W_s = \frac{1}{\lambda} L_s = \frac{1}{5}\, hour$ (3M) c) E[earning per day] = E[number of jobs done per day] $\times$ earning per job = Rs.40/- (4M)
4	**A T.V repairman finds that the time spent on his job has an exponential distribution with mean 30 minutes. If he repairs cars in the order in which they come, which follow a poisson arrival pattern with average rate of 10 per 8 hour day.** **i. What is the repairman's expected idle time each day?** **ii. How many cars are ahead of an average car brought in?** **iii. What is the average number of cars in a non- empty queue? (MAY/JUNE2012, NOV DEC 2013)(16M)**BTL5

Answer: Page : 4.24 - Dr. G. Balaji
A T.V repairmen – single server
Sets –infinite capacity

The given problem is (M/M/1) : (∞/FIFO) model

Mean arrival rate (λ)= 10 per (8 hour) day
Mean service rate (μ) = $\frac{\mathbf{8}}{\mathbf{1/2}}$ = 16 sets per (8 hour) day. (4M)

$$\rho = \frac{\lambda}{\mu} = \frac{5}{8}$$

$$P_0 = 1 - \rho = 1 - \frac{5}{8} = \frac{3}{8}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{5}{3}$$

$$L_q = L_s - \rho = \frac{25}{24} = 1.042 \qquad \text{(5M)}$$

$$W_s = \frac{1}{\lambda} L_s = \frac{1}{6}$$

$$W_q = \frac{1}{\lambda} L_q = 0.10$$

i. P[repairman id idle] = $P_0 = 1 - \rho = 1 - \frac{5}{8} = \frac{3}{8}$. (3M)

ii. Average number of jobs ahead of an average set brought in $L_s = \frac{\rho}{1-\rho} = \frac{5}{3}$. (2M)

iii. Average number of jobs in a non-empty queue = $L_W = \frac{L_q}{\rho^2}$ = 2.667. (2M)

5 **There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour:**

1) **What fraction of the time all the typists will be busy?**
2) **What is the average number of letters waiting to be typed?**
3) **What is the average time a letter has to spend for waiting and for being typed?**
4) **What is the probability that a letter will take longer than 20 min. waiting to be typed and being typed?(NOV/DEC 2004, 2010, 2011, MAY/JUNE 2007,2009,2012,2013)**(16M) BTL5

Answer: Page : 4.56 - Dr. G. Balaji
Typists – Multiple Server
Letters – infinite capacity

$the \quad given \quad problem \quad is (M/M/s) \; : \; (\infty / FIFO) \bmod el$

$mean \quad arrival \quad rate \quad (\lambda) = 15\,per \quad hour$

$mean \quad service \quad rate \quad (\mu) = 6\,per \quad hour$

$s = 3$

$$\frac{\lambda}{\mu} = \frac{15}{6} = 2.5$$

$$\rho = \frac{\lambda}{s\mu} = 0.83$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \right]^{-1} = [6.625 + 15.32]^{-1} = 0.046$$

$$L_s = \frac{1}{ss!}\frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{(1-\rho)^2} P_0 + \frac{\lambda}{\mu} = 5.95 \cong 6$$

$$L_q = L_s - \frac{\lambda}{\mu} = 6 - 2.5 = 3.5$$

$$W_s = \frac{1}{\lambda} L_s = 0.4h$$

$$W_q = \frac{1}{\lambda} L_q = 0.2333$$

 (6M)

$$P[N \geq s] = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!(1-\rho)} \Rightarrow P[N \geq 3] = 0.70$$

1) P(all the typists are busy) = $P[N \geq 3] = 0.70$. (3M)
2) The average number of letters waiting to be typed $L_q = L_s - \frac{\lambda}{\mu} = 6 - 2.5 = 3.5$. (3M)
3) The average time a letter has to spend for waiting and for being typed = $W_s = \frac{1}{\lambda} L_s = 0.4h$ =24min. (2M)
4) $$P(W > t) = e^{-\mu t}\left\{ 1 + \frac{\left(\frac{\lambda}{\mu}\right)^s \left[1 - e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)} P_0 \right\}$$ (2M)

$$P(W > 20\,\text{min}) = P(W > \frac{1}{3}hr) = 0.4616.$$

6	**A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 min. and cars arrive for service in a Poisson process at the rate of 30 cars per hour.** **1) What is the probability that an arrival would have to wait in line?** **2) Find the average waiting time, average time spent in the system and the average number of cars in the system.** **3) For what % of time would a pump be idle on an average?(MAY/JUNE 2010,2011,2012, NOV/DEC 2018)(16M)**BTL5 **Answer: Page :4.52- Dr. G. Balaji** Petrol pumps – multiple server Cars – infinite capacity s = 4 $the \quad given \quad problem \quad is\,(M/M/s) \quad : \quad (\infty/FIFO)\,\mathrm{mod}\,el$ $mean \quad arrival \quad rate \quad (\lambda) = 30\,per \quad hour$ $mean \quad service \quad rate \quad (\mu) = \frac{1}{6}\,per \quad \min = 10\,per \quad hour$ $s = 4$ $\frac{\lambda}{\mu} = \frac{30}{10} = 3$ $\rho = \frac{\lambda}{s\mu} = \frac{30}{4(10)} = 0.75$ $1-\rho = 0.25$ $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)}\right]^{-1} = [13+13.5]^{-1} = 0.0377$ $L_s = \frac{1}{ss!}\frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{(1-\rho)^2}P_0 + \frac{\lambda}{\mu} = 4.5269$ $L_q = L_s - \frac{\lambda}{\mu} = 4.53 - 3 = 1.53$ $W_s = \frac{1}{\lambda}L_s = 0.151h = 9.06\min$ $W_q = \frac{1}{\lambda}L_q = 0.0.51h = 3.06\min$ (6M) $P[N \geq s] = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!(1-\rho)} \Rightarrow P[N \geq 4] = 0.509$

1) P [an arrival has to wait] =P [W>0] = $P[N \geq 4] = 0.509$. (2M)

2) (a) The average waiting time in the queue $= W_q = \frac{1}{\lambda} L_q = 0.0.51h = 3.06\min$. (2M)

(b) The average time spend in the system = $W_s = \frac{1}{\lambda} L_s = 0.151h = 9.06\min$. (2M)

(c) The average number of cars in the system= $L_s = \frac{1}{ss!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{(1-\rho)^2} P_0 + \frac{\lambda}{\mu} = 4.5269$ =4.53 cars.

3) The fraction of time when the pumps are busy $= 1 - \rho = 0.25$ =25%. (4M)

7

Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour. (1) find the effective arrival rate at the clinic. (2) what is the probability that an arriving patient will not wait? (3) What is the expected waiting time until a patient is discharged from the clinic?(MAY/JUNE 2007, 2010, 2012, NOV/DEC 2009)(16M)BTL5

Answer: Page : 4.75- Dr. G. Balaji

Clinic – Single server

15 Patients – Finite Capacity

Hence, this problem comes under the model (M/M/1) : (k/FIFO) (3M)

$$\begin{aligned} &Mean \quad arrival \quad rate \quad (\lambda) = 30\, per \quad hour \\ &Mean \quad service \quad rate \quad (\mu) = 20\, per \quad hour \\ &k = 14 + 1 = 15 \\ &\rho = \frac{\lambda}{\mu} = \frac{30}{20} = 1.5 \\ &here \quad \lambda \neq \mu \\ &P_0 = \frac{1-\rho}{1-\rho^{k+1}} = 0.00076 \approx 0.001 \\ &\lambda' = \mu(1-P_0) = 19.98 \approx 20 \\ &L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} = 13.02 \\ &L_q = L_s - \frac{\lambda'}{\mu} = 12.02 \\ &W_s = \frac{1}{\lambda'} L_s = 0.651 \\ &W_q = \frac{1}{\lambda'} L_q = 0.601 \end{aligned}$$

(6M)

	(1) The effective arrival rate = 20 per hour. (2M) (2) P(a patient will not wait) = P_0 = 0.001. (2M) (3) To find W_s = 0.65 h = 39 min. (3M)
8	**At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals and exponential service distribution, find the probabilities for the numbers of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results are modified?(16M)**BTL5 **Answer: Page : 4.80- Dr. G. Balaji** One yard – single server Trains – finite capacity $Mean \quad arrival \quad rate \quad (\lambda) = 6\, per \quad hour$ $Mean \quad service \quad rate \quad (\mu) = 6\, per \quad hour$ $k = 3$ $\rho = \frac{\lambda}{\mu} = \frac{6}{6} = 1$ $here \quad \lambda = \mu$ $P_0 = \frac{1}{k+1} = \frac{1}{4} = 0.25$ $\lambda' = \mu(1 - P_0) = 4.5$ $L_s = \frac{k}{2} = 1.5$ $W_s = \frac{1}{\lambda'} L_s = \frac{1}{3} h = 20 \min$ $Now, \quad average \quad number \quad of \quad trains \quad in \quad the \quad railway \quad station \quad is\, (L_s) = 1.5$ $Average \quad waiting \quad time \quad in \quad the \quad station \quad of \quad the \quad new \quad train \quad coming \quad into$ $the \quad yard \quad is\, (W_s) = 20 \min$ (8M)

$$\text{If the handling rate is doubled,}$$

$$\lambda = 6$$

$$\mu = 12$$

$$k = 3$$

$$\rho = \frac{\lambda}{\mu} = \frac{6}{12} = \frac{1}{2} = 0.5$$

$$\text{Here} \quad \lambda \neq \mu$$

(8M)

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = 0.533$$

$$\lambda' = \mu(1-P_0) = 5.604$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} = 0.733$$

$$W_s = \frac{1}{\lambda'} L_s = 7.86 \min$$

9

Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 5 trains, find the probability that the yard is empty and the average number of trains in the system, given that the inter arrival time and service time are following exponential distribution. (MAY/JUNE 2011,2012)(10M)BTL5

Answer: Page : 4.84 - Dr. G. Balaji

One yard – single server

Trains- finite capacity

Hence this problem comes under the model (M/M/1) : (k/FIFO)model (2M)

$$\text{Mean arrival rate } (\lambda) = \frac{1}{15} \text{ per } \min$$

$$\text{Mean service rate } (\mu) = \frac{1}{33} \text{ per } \min$$

$$k = 5$$

$$\rho = \frac{\lambda}{\mu} = \frac{33}{15} = 2.2$$

$$\text{here} \quad \lambda \neq \mu$$

$$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = 0.011$$

$$\lambda' = \mu(1-P_0) = 0.03$$

$$L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-\rho^{k+1}} = 4.22$$

The probability of yard is empty = P_0=0.011.

Average number of trains in the system = L_s= 4.22. (8M)

10 **A two person barber shop has 5 chairs to accommodate waiting customers. Potential**

	customers, who arrive when all 5 chairs are full leave without entering barber shop. Customers arrive at the average rate of 4 per hour and spend an average of 12 min. in the barber's chair compute P_0, P_1, P_7, $E(L_q)$ and $E(W)$. (NOV/DEC 2013)(16M)BTL5 **Answer: Page : 4.92- Dr. G. Balaji** 2 Person barber shop – multiple server Chairs – finite capacity Hence, this problem comes under the model (M/M/s): (k/FIFO) (2M) Here, s=2 K=2+5 =7 $\lambda = 4\ per\ \ hour$ $\mu = 5\ per\ \ hour$ $\frac{\lambda}{\mu} = \frac{4}{5} = 0.8$ $\rho = \frac{\lambda}{\mu s} = 0.4$ $to\ \ find\ \ P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\left(\frac{\lambda}{\mu}\right)^s\sum_{n=s}^{k}\rho^{(n-s)}\right)\right]^{-1} = 0.429$ $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, n \le s$ $\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)P_n\right]$ $here\ \ s = 2$ $\lambda' = 5[2-(2P_0+P_1)]$ $P_1 = 0.343$ $\lambda' = 3.994$ $L_s = \frac{P_0}{s!}\left(\frac{\lambda}{\mu}\right)^s\left[\frac{\rho(1-\rho^{k-s})}{(1-\rho)^2} - \frac{(k-s)\rho^{k-s+1}}{1-\rho}\right] + \frac{\lambda'}{\mu} = 0.9452$ $L_q = L_s - \frac{\lambda'}{\mu} = 0.15 customer$ $W_s = \frac{L_s}{\lambda'} = 14.20\min$ (14M) $W_q = \frac{L_q}{\lambda'} = 0.34$ $s = 2, n = 7, k = 7$ $P_7 = 0.0014.$
11	**Explain Morkovian Birth Death process and obtain the expressions for steady state probabilities.(APR/MAY 2015) (16M)**BTL5 **Answer: Page : 4.8 - Dr. G. Balaji**

	Let N(t) denotes the total number of individuals at approach '*t*' starting from *t=0*. Consider the interval 0 to *t+h*. Suppose this is split into 2 periods 0 to *t* and *t+h*. (3) A_{ij} : *(n-i+j)* individuals by approach *t, i* – birth and *j* – death between *t &t+h, i,j=0,1.* (2) $P_n(t) = P[N(t) = n]$ (2) $P_n(t+h) = P_n(t)[1-(\lambda_n + \mu_n)h] + P_{n-1}(t)[\lambda_{n-1}h] + P_{n+1}(t)[\mu_{n-1}(h)] + O(h)$ (3) $as \quad h \to 0 \quad we \quad have$ $P_0^{\ 1}(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$ (2) If at approach t=0 there were *i* individuals, then the initial condition is $P_n(0) = 0$, for $n \neq 1$, (2) $P_1(0) = 1$ (2) Its known as equation of birth and death process.
12	**Customers arriving at a watch repair shop according to Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes.** **(i) Find the average number of customers Ls in the shop.** **(ii) Find the average time a customer spends in the shop.** **(iii) Find the average number of customer in the queue.** **(iv) What is the Probability that the server is idle? (NOV/DEC 2005,2010)(16M)**BTL5 **Answer: Page : 4.21 - Dr. G. Balaji** The watch repair shop – single server Customer – infinite capacity The given problem is (M/M/1) : (∞/FIFO) model Mean arrival rate (λ)= $\frac{1}{10}$ customers per min. Mean service rate (μ) = $\frac{1}{8}$ per min. (2M) $\rho = \frac{\lambda}{\mu} = \frac{4}{5}$ $P_0 = 1-\rho = 1-\frac{4}{5} = \frac{1}{5}$ $L_s = \frac{\rho}{1-\rho} = 4$ $L_q = L_s - \rho = \frac{16}{5} = 3.2$ (6M) $W_s = \frac{1}{\lambda} L_s = 40$ $W_q = \frac{1}{\lambda} L_q = 32$

	(i) $L_s = \frac{\rho}{1-\rho} = 4$ customers (2M) (ii) $W_s = \frac{1}{\lambda} L_s = 40$ min. (2M) (iii) $L_q = L_s - \rho = \frac{16}{5} = 3.2$ customers. (2M) **(v)** P [server is idle] = $\frac{1}{5}$. (2M)
13	**A supermarket has a single cashier. During peak hours, customers arrive at a rate of 20 per hour. The average number of customers that can be proceed by the cashier is 24 per hour. Calculate** **1) The probability that the cashier is idle.** **2) The average number of customers in the Queuing system.** **3) The average time a customer spends in the system,** **4) The average number of customers in the queue.** **5) The average time a customer spends in the queue waiting for service.(APR/MAY 2014)(16M)**BTL5 **Answer: Page : 4.19 - Dr. G. Balaji** Single cashier – single server Customers – infinite capacity. The given problem is (M/M/1) : (∞/FIFO) model. (3M) Mean arrival rate (λ)= 20 per hour. Mean service rate (μ) =24 per hour. $\rho = \frac{\lambda}{\mu} = \frac{20}{24}$. (3M) $P_0 = 1 - \rho = 1 - \frac{20}{24} = \frac{4}{24}$ $L_s = \frac{\rho}{1-\rho} = 5$ $L_q = L_s - \rho = \frac{25}{6} = 4.1667$ (4M) $W_s = \frac{1}{\lambda} L_s = \frac{1}{4} h$ $W_q = \frac{1}{\lambda} L_q = 0.2083h$ 1) The probability that the cashier is idle: $P_0 = 1 - \rho = 1 - \frac{20}{24} = \frac{4}{24}$. (2M) 2) The average number of customers in the system = $L_s = \frac{\rho}{1-\rho} = 5$. (2M) 3) The average time a customer spends in the system = $W_s = \frac{1}{\lambda} L_s = \frac{1}{4} h$

	4) The average number of customers waiting in the queue = $L_q = L_s - \rho = \frac{25}{6} = 4.1667$ 5) The average time a customer spends in the queue = $W_q = \frac{1}{\lambda} L_q = 0.2083h$
14	**A supermarket has 2 girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 min and if people arrive in Poisson fashion at the rate of 10 per hour,** **(1) What is the probability that a customer has to wait for service?** **(2) What is the expected % of idle time for each girl?** **(3) If the customer has to wait in the queue, what is the expected length of his waiting time? (APR/MAY 2011,2015)**BTL5 **Answer: Page : 4.58– G. Balaji** Girls – multiple server People – infinite capacity $the \quad given \quad problem \quad is (M/M/s) \quad : \quad (\infty/FIFO) \bmod el$ $mean \quad arrival \quad rate \quad (\lambda) = 10 \, per \quad hour = \frac{1}{6} \, per \quad \min$ $mean \quad service \quad rate \quad (\mu) = \frac{1}{4} \, per \quad \min$ $s = 2$ $\frac{\lambda}{\mu} = \frac{1/6}{1/4} = 0.67$ $\rho = \frac{\lambda}{s\mu} = 0.33$ $1 - \rho = 0.67$ $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n + \frac{\left(\frac{\lambda}{\mu} \right)^s}{s!(1-\rho)} \right]^{-1} = [1.67 + 0.355]^{-1} = 0.5$ (1) P[a customer has to wait] = $P[N \geq s] = \frac{\left(\frac{\lambda}{\mu} \right)^s P_0}{s!(1-\rho)} \Rightarrow P[N \geq 2] = 0.168$ (2) The fraction of time when the girls are busy = $\frac{\lambda}{s\mu} = \frac{1}{3}$. (3) $E(W_q / W_s > 0) = \frac{1}{\mu s - \lambda} = 3 \min$.
15	**Derive the governing equations for the (M/M/1) : (GD/N/∞) queuing model and hence obtain the expression for the steady state probability and the average number of customers**

in the system.
(or)
Derive the steady- state probability of the number of customers in M/M/1queueing system from the birth and death processes and hence deduce that the average measures such as expected system size, expected queue size, expected waiting time in system and expected waiting time in queue. (NOV/DEC 2011,2013, NOV/DEC 2018) (16M) BTL5
Answer: Page : 4.15 - Dr. G. Balaji
Let 'N' denotes the number of customers in the queuing system and the number of customer in the queue is (N-1)
λ – Mean arrival rate
μ- Mean service rate
ρ- Traffic intensity $=\frac{\lambda}{\mu}$
$P_0 = 1-\rho$;
$P_n = \rho^n(1-\rho), \rho<1, n=0,1.2....$

(1) Average number of customers in the system: $L_s = \frac{\rho}{1-\rho}$
(2) Average number of customer in the queue: $L_q = L_s - \rho$
(3) The Average waiting time of a customer in the system: $W_s = \frac{1}{\lambda}L_s$
(4) The average waiting time of a customer in the queue: $W_q = \frac{1}{\lambda}L_q$
(5) Average number of customer in non-empty queues: $L_W = \frac{L_q}{P(n>1)} = \frac{L_q}{\rho^2}$
(6) The probability density function of the waiting time in the system:
$$f(W) = \mu\left(\frac{\mu-\lambda}{\mu}\right)e^{-\mu W}e^{\lambda W} = (\mu-\lambda)e^{-(\mu-\lambda)}W$$
Which is the p.d.f of an exponential distribution with parameter $(\mu-\lambda)t$.

	UNIT V - ADVANCED QUEUEINGMODELS
	Finite source models - M/G/1 queue – Pollaczek Khinchin formula - M/D/1 and M/E$_K$/1 as special cases – Series queues – Open Jackson networks.
Q.No	**Part * A**
1.	**Write down Pollaczek- Khintchine formula and explain the notation.(NOV/DEC 2011,2013)**BTL1 If T is the random service time, the average number of customers in the system $L_s = E_n = \lambda E(T) + \frac{\lambda^2[E^2(T)+V(T)]}{2[1-\lambda E(T)]}$ Where E(T) is mean of T and V(T) is variance of T.
2	**M/G/1 queuing system is Markovian. Comment on this statement.**BTL2 M/G/1 queuing system is a non-Markovian queue model. Since the service time follows general distribution. In the M/G/1queuing system under study, we consider a single-server queuing system with infinite capacity, Poisson arrivals and general service discipline. The model has arbitrary service time, and it is not necessary to be memoryless (i.e) it is not exponential.
3	**Write down the Pollaczek – Khintchine transform formula**.BTL1 The Pollaczek- Khintchine Transform formula: $V(s) = \frac{(1-\rho)(1-s)B^*(\lambda-\lambda_S)}{B^*(\lambda-\lambda_S)-s}$
4	**In M/G/1 model write down the formula for the average number of customers in the system.**BTL1 The average number of customers in the system is $W_s = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$
5	**Write the classification of Queuing Networks.(MAY/JUNE 2010)**BTL1 1) Open Networks. 2) Closed Networks. 3) Mixed Networks.
6	**State Arrival theorem. (MAY/JUNE 2010)**BTL1 In the closed network system with *m* customers, the system as seen by arrivals to server *j* is distributed as the stationary distribution in the same network system when there are only *m-1* customers.
7	**Distinguish between open and closed network.(APR/MAY 2010,2011,2014,NOV/DEC 2015)**BTL2

	Open Network: Arrivals from outside to the node i is allowed. Once a customer gets the service completed at node i, he joins the queue at node j with probability P_{ij} or leaves the system with Probability P_{i0}. Closed Network: New customers never enter in to the system. Existing customers never depart from the system (i.e)., $P_{i0} = 0$ and $r_i = 0$ for all i (OR)No customer may leave the system.
8	**Explain (Series queue) tandem queue model.(NOV/DEC 2010,2011)**BTL2 A series queue model or a tandem queue or a tandem queue model is satisfies the following characteristics. 1) Customers may arrive from outside the system at any node and may leave the system from any node. 2) Customers may enter the system at some node, traverse from node to node in the system and leave the system from some node, necessarily following the same order of nodes. 3) Customers may return to the nodes already visited, skip some nodes and even choose to remain in the system forever.
9	**Define an open Jackson network. (APR/ MAY 2015, NOV/DEC 2013, 2014)**BTL1 Suppose a queuing network consists of k nodes is called an open Jackson network, if it satisfied the following characteristics. 1) Customers arriving at node k from outside the system arrive in a Poisson pattern with the average arrival rate r_i and join the queue at i and wait for his turn for service. 2) Service times at the channels at node i are independent and each exponentially distributed with parameter μ . 3) Once a customer gets the service completed at node i, he joins the queue at node j with probability P_{ij} when $i=1, 2,\ldots,k$ and $j=0,1,2,\ldots k$. P_{i0}represents the probability that a customer leaves the system from node i after getting the service at i. 4) The utilization of all the queues is less than one.
10	**What is meant by queue network?** BTL1 A network of queues is a collection of service centers, which represent system resources, and customers, which represent users or transaction.
11	**Define Closed queuing network.(MAY/JUNE 2013)** BTL1 In a closed queuing network, jobs neither enter nor depart from the network. If the network has multiple job classes then it must be closed for each class of jobs.
12	**Define Open queuing network.(APR/MAY 2015)** BTL1 An open queuing network is characterized by one or more sources of job arrivals and corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes then it must be open for each class of jobs.
13	**What do you mean by bottleneck of a network? (NOV/DEC 2010)**BTL2 As the arrival rate λ in a 2-state tandem queue model increases, the node with the larger value of $\rho_i = \frac{\lambda}{\mu_i}$ will introduce instability. Hence the node with the larger value ρ_i is called the bottleneck of the

	system.
14	**Consider a service facility with two sequential stations with respective service rate of 3/min and 4/min. The arrival rate is 2/min. What is the average service time of the system, if the system could be approximated by a two stage Tandem queue? (NOV/DEC 2010)**BTL3 $\lambda = 2$ $\mu_1 = 3$ Given $\mu_2 = 4$ $\text{The average service time of the system} = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = 1 + \frac{1}{2} = \frac{3}{2} / \min$.
15	**What do you mean by series queue with blocking?(APR/MAY 2011)**BTL2 This is a sequential queue model consisting of two service points S_1 and S_2 at each of which there is only one server and where no queue is allowed to form at either point.
16	**Define a two Stage tandem queues. (APR/MAY 2011)**BTL1 Consider a two- server system in which customers arrive at a Poisson rate λ at server 1. After being served by server 1 then they join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server one customer at a time with server *i* taking an exponential time with rate μ_i for service *i=1,2,…* such a system is called a tandem or sequential system.
17	**Write down the balance equation for 2- stage series queue model.**BTL1 $\lambda p(0,0) = \mu_2 p(0,1)$ $(\lambda + \mu_1) p(m,0) = \lambda p(m-1,0) + \mu_2 p(m,1), [m > 0]$ $(\lambda + \mu_2) p(0,n) = \lambda p(1,n-1) + \mu_2 p(0,n+1), [n > 0]$ $(\lambda + \mu_1 + \mu_2) p(m,n) = \lambda p(m-1,n) + \mu_1 p(m+1,n-1) + \mu_2 p(m,n+1), [m > 0]$ $\sum_m \sum_n p(m,n) = 1$
18	**Write down the (flow balance) traffic equation for an open Jackson network.(MAY/JUNE 2016)**BTL1 Jackson's flow balance equation for this open model are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij},\ j = 1,2,..,k$
19	**Given any two examples for series queuing situation. (APR/MAY 2015)**BTL2 1) A master health check-up programme in a hospital where a patient has to undergo a series of test. 2) An admission process in a school where the student has to visit a series of officials. 3) Manufacturing or assembly line process. 4) Registration process in university. 5) Clinic physical examination procedure.
20	**Define a Tandem Queue**. BTL1 A series queue in which the series facilities are arranged in sequence and the flow is always in a single

	direction.
21	**When a M/G/1 queuing model will become a classic M/M/1 queuing model?(MAY/JUNE 2012)**BTL2 In the M/G/1 model, G stands for the general service time distribution. If G is replaced by exponential service time distribution then the M/G/1 model become the classic M/M/1 model.
22	**Consider a tandem queue with 2 independent Markovian servers. The situation at server 1 is just as in an M/M/1model. What will be the type of queue in server 2? Why?**BTL2 The type of queue in server 2 is also a M/M/1 model. Since output of M/M/1 is another M/M/1 queue.
23	**Define series queues.(NOV/DEC 2013)**BTL1 A series queue is one in which customers may arrive from outside the system at any node and may leave the system from any node.
24	**What does the letter in the symbolic representation M/G/1 of a queuing model representation M/G/1 of a queuing model represent?(APR/MAY 2015)**BTL1 M- Inter arrival time is exponential distribution. G- Service time is general distribution 1-Number of server.
25	**How queuing theory could be used to study computer network. (APR/MAY 2010)**BTL2 1) Jackson's open network concept can be extended when the nodes are multi server nodes. In this case the network behaves as if each node is an independent M/M/S model. 2) Consider a system of k servers. Customers arrive from outside the system to server i, $i=1,2,3\ldots k$ in accordance with independent Poisson processes then they join the queue at i until their turn at service comes. Once a customer is served by server i, then he joins the queue in front of server j, $j=1, 2, \ldots, k$ with probability P_{ij}. Hence $\sum_{j=1}^{k} P_{ij} \leq 1$ and 1-$\sum_{j=1}^{k} P_{ij}$ represents the probability that a customer departs the system after being served by server i. if we let λ_j denote the total arrival rate of customers to server j, then the λ_j can be obtained as the solution of $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}, j = 1,2,..,k$.
	Part * B

1

Derive Pollaczek – Khintchine formula.

(OR)

Derive Pollaczek – Khintchine formula for M/G/1 queue. Hence deduce the result for the result for the queues M/D/1 and M/E$_K$/1 as special cases. (APR/MAY 2010,2011, NOV/DEC 2010,2011,2012, 2013,2014,2015, NOV/DEC 2018) (16M) BTL5

Answer: Page : 5.2 - Dr. G. Balaji

Let $n' = n - 1 + \delta + k$

n →Number of customers in the system at time$'t'$.

n' →number of customers in the system $'t + T'$

T → random service time

k →Number of arrivals during the service time.

$$\delta = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n > 0 \end{cases}$$

$$n' = \begin{cases} k & if\ n = 0 \\ (n-1) + k & if\ n > 0 \end{cases}$$

$\delta^2 = \delta$ [for values of $\delta = 0$ and $\delta = 1$]

$$n\delta = 0 \quad [\because n = 0, \delta = 1 \Rightarrow n\delta = 0; n > 0, \delta = 0 \Rightarrow n\delta = 0]$$

$$n' = n + \delta - (1 - k)$$

$$E\left[n'^2\right] - E[n^2] + 2E[n][1 - E(k)] = -E[\delta] + 1 - E[k] + E[k^2] - E[k] + 2E[\delta]E[k]$$

$$E[n] = E(k) + \frac{E[k^2] - E[k]}{2[1 - E(k)]} \qquad (6M)$$

$$E[k] = E[\lambda T] = \lambda E[T]$$

$$E[n] = \lambda E[T] + \frac{\lambda^2 V(T) + \lambda^2 [E(T)]^2}{2[1 - \lambda E(T)]}$$

$$L_s = \rho + \frac{\rho^2}{2(1 - \rho)}$$

M → arrival time follows Poisson distribution

E_k → service time follows Erlang distribution with k phases

1→ Siingle server model

Here, $\mu = \frac{1}{km}$, $\sigma^2 = \frac{1}{k\mu^2}, \rho = \frac{\lambda}{\mu}$

	Hence, from P-K formula, we get $L_s = \frac{\lambda}{\mu} + \left(\frac{k+1}{2k}\right)\frac{\lambda^2}{\mu(\mu-\lambda)}$. (10M)
2	**In a heavy machine shop, the overhead crane is 75% utilized. Time study observation gave the average slinging time as 10.5 min with a standard deviation of 8.8 min.** **(1) What is the average calling rate for the service of the crane?** **(2) What is the average delay in getting service?** **(3) If the average service time is cut to 8.0 min, with a standard deviation of 6.0 min, how much reduction will occur, on average, in the delay of getting served? (16M)** BTL3 **Answer: Page : 5.6 - Dr. G. Balaji** This is a (M/G/1) : (∞/FIFO) Process Utilization rate = 75%= $\frac{3}{4}$ $\rho = \frac{\lambda}{\mu} = \frac{3}{4}$ Mean service time = 10.5 min $\lambda = \frac{3}{4}\mu$ λ= 0.0714 per min $\mu = \frac{1}{10.5}$ $\rho = 0.75$ $\sigma = 8.8$ $L_s = \rho + \frac{\rho^2}{2(1-\rho)}$ = 2.6646 $L_q = L_s - \frac{\lambda}{\mu} = 1.9146$ $W_s = \frac{1}{\lambda} L_s = 37.32$ (10M) $W_q = \frac{1}{\lambda} L_s = 26.815$ (1) The average calling rate for the services of the crane = λ= 0.0714 per min. (2M)

	(2) The average delay in getting service = $W_q = \frac{1}{\lambda} L_s = 26.815$. (2M) (3) The reduction will occur on average, in the delay of getting served = 26.815 - 8.325 = 18.5 min. (2M)
3	**In a big factory, there are a large number of operating machines and two sequential repair shops, which do the service of the damaged machines exponentially with respective rates of 1/hour and 2/hour. If the cumulative failure rate of all the machines in the factory is 0.5/hour, find (i) the probability that both repair shops are idle, (ii) the average number of machines in the service section of the factory and (iii) the average repair time of a machine. (NOV/DEC 2010) (10M)** BTL3 **Answer: Page : 5.49 - Dr. G. Balaji** $\lambda = 0.5 / hour = \frac{1}{2}\ per\ \ hour$ $\mu_1 = 1\ per\ \ hour$ $\mu_2 = 2\ per\ \ hour$ $P(both\ \ the\ \ service\ \ stations\ \ are\ \ idle)$ $P(0,0) = \left(\frac{\lambda}{\mu_1}\right)^0 \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^0 \left(1 - \frac{\lambda}{\mu_2}\right) = \frac{3}{8}$ (5M) $The\ \ average\ \ number\ \ of\ \ machines\ \ in\ \ service$ $= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} = \frac{4}{3}$ $The\ \ average\ \ repair\ \ time = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{8}{3}$ (5M)
4	**A TVS company in Madurai containing repair facility shared by a large number of machines has 2 sequential stations with respective rates of 2per hour and 3per hour. The cumulative failure rate of all the machines is 1 per hour. Assuming that the system behavior may be approximated by the 2-stage tandem queue, find** **(1) the average repair time including the waiting time.** **(2) the probability that both the service stations are idle** **(3) the bottleneck of the repair facility.** **(OR)** **A repair facility shared by a large number of machines has 2- sequential stations ith respective service rates of 2 per hour and 3 per hour. The cumulative failure rate of all the machines is 1 per hour. Asuming that the system behavior may be approximated by the 2-stage tandem queue, find** **(1) The average repair time including the waiting time,** **(2) The probability that both the service stations are idle and** **(3) The bottleneck of the repair facility. (APR/MAY 2015) (10M)** BTL3 **Answer: Page : 5.15 - Dr. G. Balaji**

$\lambda = 1$

$\mu_1 = 2$

$\mu_2 = 3$

(1)*The average number of machines in service* (5M)

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda}{\mu_2 - \lambda} = \frac{3}{2}$$

$$(2)\ The\ \ average\ \ repair\ \ time = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{8}{3}$$

$$(3) P(0,0) = \left(\frac{\lambda}{\mu_1}\right)^0 \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^0 \left(1 - \frac{\lambda}{\mu_2}\right) = \frac{3}{8}$$

(5M)

5

An average of 120 students arrive each hour (inter arrival times are exponential) at the controller office to get their hall tickets. To complete the process, a candidate must pass through three counters. Each counter consists of a single server; service times at each counter are exponential with the following mean times: counter 1, 20 seconds; counter 2, 15 seconds and counter 3, 12 seconds. On the average how many students will be present in the controller's office. (MAY/JUNE 2012, APR/MAY 2014)(8M) BTL3

Answer: Page : 5.61- Dr. G. Balaji

$$\lambda = 120 / hr$$

$$\mu_1 = \frac{1}{20} / \sec = 180 / hr$$

$$\mu_2 = \frac{1}{15} / \sec = 240 / hr$$

$$\mu_3 = \frac{1}{12} / \sec = 300 / hr$$

$$L_{s1} = \frac{\lambda}{\mu_1 - \lambda} = 2$$

(4M)

$$L_{s2} = \frac{\lambda}{\mu_2 - \lambda} = 1$$

$$L_{s3} = \frac{\lambda}{\mu_3 - \lambda} = \frac{2}{3}$$

$$Average\ \ number\ \ of\ \ students = L_{s1} + L_{s2} + L_{s3} = \frac{11}{3}$$

(4M)

6	**Consider a system of two servers where customers from outside the system arrive at server 1 at a Poisson rate 4 and at server 2 at a Poisson rate 5. The service rates 1 and 2 are respectively 8 and 10. A customer upon completion of service at server 1 is equally likely to go to server 2 or to leave the system (i.e., $P_{11} = 0$, $P_{12} = ½$); whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise (i.e., $P_{21} = 1/4$, $P_{22} = 0$). Determine the limiting probabilities, Ls and Ws. [MAY/JUNE 2013] (8M)** BTL3 **Answer: Page : 5.65 - Dr. G. Balaji** $r_1 = 4; r_2 = 5$ $\mu_1 = 8; \mu_2 = 10$ *The Jackson's flow balance equations are* $\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i P_{ij}, j = 1,2$ *For* $j = 1$ *we get* $\lambda_1 = 4 + \frac{\lambda_4}{4}$ *For* $j = 2$ *we get* $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22}$ $\Rightarrow \lambda_1 = 6; \lambda_2 = 8.$ $L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} = 3 + 4 = 7.$ $W_s = \frac{1}{\lambda} L_s = \frac{7}{9}. \quad [\because \lambda = 4 + 5]$ (8M)
7	**Consider two servers. An average of 8 customers per hour arrive from outside at server 1 and an average of 17 customers per hour arrive from outside at server 2. Inter arrival times are exponential. Server 1 can serve at an exponential rate of 20 customers per hour and server 2 can serve at an exponential rate of 30 customers per hour. After completing service at server 1, half of the customers leave the system, and half go to server 2. After completing service at server 2, ¾ of the customers complete service, and ¼ return to server 1. (i) What fraction of the time is server 1 idle? (ii) Find the expected number of customers at each server. (iii) Find the average time a customer spends in the system. (iv) How would the answers to parts (i) – (iii) change if server 2 could server only an average of 20 customers per hour? [NOV/DEC 2012, 2014] (8M)** BTL3 **Answer: Page : 5.70 - Dr. G. Balaji**

	$r_1 = 8; r_2 = 17;$ $\mu_1 = 20; \mu_2 = 30$ *The Jackson's flow balance equations are* $\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i P_{ij}, j = 1,2$ *For* $j = 1$ *we get* $\lambda_1 = 8 + \frac{\lambda_2}{4}$ *For* $j = 2$ *we get* $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22}$ $\Rightarrow \lambda_1 = 14; \lambda_2 = 24,$ $(i) P_0 = 1 - \rho = 1 - \left(\frac{\lambda}{\mu}\right) = 0.3$ $(ii) L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{7}{3} + 4 = \frac{19}{3}.$ $(iii) W_s = \frac{1}{\lambda} L_s = \frac{19}{75}. \quad [\because \lambda = 8 + 17 = 25].$ (8M) $(iv) S_2 \mu_2 = 20 < \lambda_2$, *so no steady state exists.*
8	**In a network of 3 service stations 1, 2, 3 customers arrive 1, 2, 3 from outside, in accordance with Poisson process having rates 5, 10, 15 respectively. The service times at the 3 stations are exponential with respective rates 10, 50, 100. A customer competing service at station 1 is equally like to (1) go to station 2, (2) go to station 3 and (3) leave the system. A customer departing from service at station 2 always goes to station 3. A departure from service at station 3 is equally like to go to station 2 or leave the system. (A) What is the average number of customers in the system consisting of all the tree stations? (B) What is the average time a customer spends in the system? [NOV/DEC 2010, 2011] (8M)** BTL3 **Answer: Page : 5.76 - Dr. G. Balaji** $r_1 = 5; r_2 = 10; r_3 = 15$ $\mu_1 = 10; \mu_2 = 50; \mu_3 = 100;$ *The Jackson's flow balance equations are* $\lambda_j = r_j + \sum_{i=1}^{3} \lambda_i P_{ij}, j = 1,2,3$ (4M) *For* $j = 1$ *we get* $\lambda_1 = 5$ *For* $j = 2$ *we get* $\lambda_2 = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22} + \lambda_3 P_{32}$

	$For \quad j=3 \quad we \quad get$ $\lambda_3 = r_3 + \lambda_1 P_{13} + \lambda_2 P_{23} + \lambda_3 P_{33}$ $\Rightarrow \lambda_1 = 5; \lambda_2 = 40, \lambda_3 = \left(\frac{170}{3}\right),$ (4M) $L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} + \frac{\lambda_3}{\mu_3 - \lambda_3} = \frac{82}{13} = 6.3077.$ $W_s = \frac{1}{\lambda} L_s = \frac{41}{195} = 0.2103.$
9	**A one man barber shop takes exactly 25 minutes to complete one haircut. If customer arrive at the barber shop in a Poisson fashion at an average rate of one every 40 minutes, how long on the average a customer spends in the shop? Also find the average time a customer must wait for service.(NOV/DEC 2013) (8M)** BTL3 **Answer: Page : 4.15 - Dr. G. Balaji** $\lambda = \frac{1}{40}$per min $\mu = \frac{1}{25}$per min $\rho = \frac{\lambda}{\mu} = \frac{5}{8}$ $L_s = \rho + \frac{\rho^2}{2(1-\rho)} = \frac{55}{48}$ (4M) $L_q = L_s - \frac{\lambda}{\mu} = \frac{25}{48}$ $W_s = \frac{1}{\lambda} L_s = 45.833$ $W_q = \frac{1}{\lambda} L_s = 20.833$ Hence, a customer has to spend 45.8 minutes in the shop and has to wait for 20.8 minutes on the average. (4M)
10	**An automatic car wash facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars/hr. and may wait in the facility's parking lot if the bay is busy. Find L_s, L_q, W_s, W_q if the service time.** **(1) Follows uniforms distribution between 8 and 12 minutes.** **(2) Follows normal distribution with mean 12 minutes and S.D 3 minutes** **(3) Follows a discrete distribution with values 4,8 and 15 minutes with corresponding probability 0.2, 0.6 and 0.2. (16M)** BTL3 **Answer: Page : 5.16 - Dr. G. Balaji** This is an M/G/1 queue model. (a) Mean $=\lambda = \frac{4}{60}$per minute. $E(T)$=mean of the uniform distribution $=\frac{1}{2}(a+b)$=10

	$Var(T) = \frac{1}{12}(b-a)^2 = \frac{4}{3}$. By the Pollazek- Knichine formula, $L_s = \frac{302}{225} = 1.342$ cars. $L_q = 0.675$ cars $\cong 1$ car. (by Little's formula) (5M) (b) Mean $= \lambda = \frac{1}{15}$. $E(T)$=12min $Var(T) = 9$. $\mu = \frac{1}{E(T)} = \frac{1}{12}$ By the Pollazek- Knichine formula, $L_s = 2.5$ cars. $L_q = 1.7$ cars $\cong 2$ cars.(by Little's formula) (5M) (C) T: 4 8 15 $P(T)$: 0.2 0.6 0.2 $E(T) = \sum TP(T) = 8.6$min $var(T) = E(T^2)$-$[E(T)]^2$ =12.64 By the Pollazek- Knichine formula, $L_s = 1.021 \cong 1$ car. $L_q = 0.45$cars (by Little's formula) (6M)
11	**Jackson network with three facilities that have the parameters given below** $P_{11} = 0, P_{12} = 0.6, P_{13} = 0.3,$ $P_{21} = 0.1, P_{22} = 0, P_{23} = 0.3,$ $P_{31} = 0.4, P_{32} = 0.4, P_{33} = 0,$ $\mu_1 = 10, \mu_2 = 10, \mu_3 = 10,$ $c_1 = 1, c_2 = 2, c_3 = 1,$ $r_1 = 1, r_2 = 4, r_3 = 3$ 1) **Find the total arrival rate at each facility** 2) **Find** $P(n_1, n_2, n_3)$ 3) **Find the expected number of customers in the entire system** 4) **Find the expected time a customer spends in the system. [MAY/JUNE 2012, APR/MAY 2014]** **(OR)** **For an open queuing network with three nodes 1, 2, and 3, let customers arrive from outside the system to node j according to a Poisson input process with parameters r_j and let P_{ij} denote the proportion of customers departing from facility i to facility j. Given (r_1, r_2, r_3) = (1, 4, 3) and $P_{ij} = \begin{pmatrix} 0 & 0.6 & 0.3 \\ 0.1 & 0 & 0.3 \\ 0.4 & 0.4 & 0 \end{pmatrix}$. Determine the average arrival rate λ_j to the node j for $j = 1, 2, 3$.** **(16M)** BTL3 **Answer: Page : 5.84 - Dr. G. Balaji**

	$P_{11}=0, P_{12}=0.6, P_{13}=0.3,$ $P_{21}=0.1, P_{22}=0, P_{23}=0.3,$ $P_{31}=0.4, P_{32}=0.4, P_{33}=0,$ $\mu_1=10, \mu_2=10, \mu_3=10,$ $c_1=1, c_2=2, c_3=1,$ $r_1=1, r_2=4, r_3=3$ $r_1=5; r_2=10; r_3=15$ $\mu_1=10; \mu_2=50; \mu_3=100;$ *The Jackson's flow balance equations are* $\lambda_j=r_j+\sum_{i=1}^{3}\lambda_i P_{ij}, j=1,2,3$ *For* $j=1$ *we get* $\lambda_1=1+(0.1)\lambda_2+(0.4)\lambda_3$ *For* $j=2$ *we get* $\lambda_2=4+(0.6)\lambda_1+(0.4)\lambda_3$ *For* $j=3$ *we get* $\lambda_3=3+(0.3)\lambda_1+(0.3)\lambda_2$ $\Rightarrow \lambda_1=5; \lambda_2=10, \lambda_3=7.5,$ *Facility 1 is an* $(M/M/1)$ *model* (10M) $P_{n1}=\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda_1}{\mu_1}\right)=\left(\frac{1}{2}\right)^{n_1}\left(\frac{1}{2}\right)$ $L_{s_1}=\frac{\lambda_1}{\mu_1-\lambda_1}=1$ Facility 2 is an (M/M/2) model $P_{n_2}=\begin{cases}\frac{1}{n_2!}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}P_0, & If\ n_2<2\\ \frac{1}{C_2!C_2^{n_2-C_2}}\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}P_0, & If\ n_2\geq 2\end{cases}$ $P_0=\left[\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{\lambda_2}{\mu_2}\right)^n+\frac{\frac{1}{2!}\left(\frac{\lambda_2}{\mu_2}\right)^2}{\left(1-\frac{\lambda_2}{2\mu_2}\right)}\right]^{-1}=\frac{1}{3}$ $P_1=\frac{1}{1!}\left(\frac{\lambda_2}{\mu_2}\right)^1 P_0=\frac{1}{3}$

	$P_{n_2} = \left\{ \begin{array}{l} \frac{1}{3} if \quad n_2 = 0 \\ \frac{1}{3} if \quad n_2 = 1 \\ \frac{1}{3}\left(\frac{1}{2}\right)^{n_2 - 1} if \quad n_2 \geq 0 \end{array} \right\}$ $L_{s_2} = \frac{\left(\frac{\lambda_2}{\mu_2}\right)^{C_2+1}}{c_2 c_2! \left(1 - \left(\frac{\lambda_2}{c_2 \mu_2}\right)\right)^2} P_0 + \frac{\lambda_2}{\mu_2} = \frac{4}{3}$ $Facility \quad 3 \quad is \quad an \quad (M/M/1) \quad \text{mod}\, el$ $P_{n_3} = \left(\frac{\lambda_3}{\mu_3}\right)^{n_3}\left(1 - \frac{\lambda_3}{\mu_3}\right) = \left(\frac{7.5}{10}\right)^{n_3}\left(\frac{2.5}{10}\right)$ $L_{s_3} = \frac{\lambda_3}{\mu_3 - \lambda_3} = 3$ $L_s = L_{s_1} + L_{s_2} + L_{s_3} = \frac{16}{3}$ (6M) $W_s = \frac{L_s}{\lambda} = \frac{2}{3}$
12	**For a 2-stage (service point) sequential queue model with blockage, compute L_s and W_s, if $\lambda = 1, \mu_1 = 1$ and $\mu_2 = 2$. (16M)** BTL3 **Answer: Class Work Note** Given $\lambda = 1, \mu_1 = 1$ and $\mu_2 = 2$ The balanced equation are (0,0) $\lambda P_{00} = \mu P_{01}$ (1,0) $\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$ (0,1) $(\lambda + \mu_1) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$ (1,1) $(\mu_1 + \mu_2) P_{11} = \lambda P_{01}$ (b,1) $\mu_2 P_{b1} = \mu_1 P_{11}$ (4M) $P_{00} = 3P_{01}$ $P_{10} = P_{00} + 3P_{11}$ $4P_{01} = P_{10} + 3P_{b1}$ $4P_{11} = P_{01}$ $3P_{b1} = P_{11}$ $P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$ (4M) From above equations we have $P_{10} = \frac{4}{3} P_{00}$ $P_{00} = \frac{12}{37}$ $P_{01} = \frac{6}{37}$

	$P_{11} = \frac{2}{37}$ $P_{b1} = \frac{1}{37}$ $P_{10} = \frac{16}{37}$ Therefore, $L = P_{01} + P_{10} + 2(P_{11} + P_{b1}) = \frac{65}{97}$. (6M) $W = \frac{L}{\lambda(P_{00}+P_{01})} = \left(\frac{65}{48}\right)$. (6M)
13.	**Explain Queuing network. (8M)** BTL2 **Answer: Class Work Note** In a closed queuing network, jobs neither enter nor depart from the network. If the network has multiple job classes then it must be closed for each class of jobs. (3M) An open queuing network is characterized by one or more sources of job arrivals and corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes then it must be open for each class of jobs.. Suppose a queuing network consists of *k* nodes is called an open Jackson network, if it satisfied the following characteristics. 1) Customers arriving at node *k* from outside the system arrive in a Poisson pattern with the average arrival rate r_i and join the queue at *i* and wait for his turn for service. 2) Service times at the channels at node *i* are independent and each exponentially distributed with parameter μ . 3) Once a customer gets the service completed at node *i*, he joins the queue at node *j* with probability P_{ij} when $i=1, 2, \ldots, k$ and $j=0,1,2,\ldots k$. P_{i0}represents the probability that a customer leaves the system from node *i* after getting the service at *i*. The utilization of all the queues is less than one. (5M)