JEPPIAAR INSTITUTE OF TECHNOLOGY
"Self-Belief | Self Discipline | Self Respect"

DNV·GL

DEPARTMENT
OF
ELECTRICAL AND ELECTRONICS ENGINEERING

LECTURE NOTES

EE8451- ELECTROMAGENTIC FIELDS

(2017 Regulation)

Year/Semester: II/IV EEE

2020-2021

Prepared by
Dr. Prajith Prabhakar
Assistant Professor / EEE

EC8451   **ELECTROMAGNETIC FIELDS**   **L T P C**
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OBJECTIVES:
- To impart knowledge on the basics of static electric and magnetic field and the associated laws.
- To give insight into the propagation of EM waves and also to introduce the methods in computational electromagnetics.
- To make students have depth understanding of antennas, electronic devices, Waveguides is possible.

UNIT I INTRODUCTION   **9**
Vector Algebra, Coordinate Systems, Vector differential operator, Gradient, Divergence, Curl, Divergence theorem, Stokes theorem, Coulombs law, Electric field intensity, Point, Line, Surface and Volume charge distributions, Electric flux density, Gauss law and its applications, Gauss divergence theorem, Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

UNIT II   ELECTROSTATICS   **9**
Conductors and dielectrics in Static Electric Field, Current and current density, Continuity equation, Polarization, Boundary conditions, Method of images, Resistance of a conductor, Capacitance, Parallel plate, Coaxial and Spherical capacitors, Boundary conditions for perfect dielectric materials, Solution of Laplace equation, Application of Poisson's and Laplace's equations.

UNIT III   MAGNETOSTATICS   **9**
Biot -Savart Law, Magnetic field Intensity, Estimation of Magnetic field Intensity for straight and circular conductors, Ampere"s Circuital Law, Point form of Ampere"s Circuital Law, Stokes theorem, Magnetic flux and magnetic flux density, The Scalar and Vector Magnetic potentials, Derivation of Steady magnetic field Laws.

UNIT IV   MAGNETIC FORCES AND MATERIALS   **9**
Force on a moving charge, Force on a differential current element, Force between current elements, Force and torque on a closed circuit, The nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions involving magnetic fields, The magnetic circuit, Potential energy and forces on magnetic materials, Inductance, Basic expressions for self and mutual inductances, Inductance evaluation for solenoid, toroid, coaxial cables and transmission lines, Energy stored in Magnetic fields.

UNIT V TIME VARYING FIELDS AND MAXWELL'S EQUATIONS   **9**
Fundamental relations for Electrostatic and Magnetostatic fields, Faraday's law for Electromagnetic induction, Transformers, Motional Electromotive forces, Differential form of Maxwell's equations, Integral form of Maxwell's equations, Potential functions, Electromagnetic boundary conditions, Wave equations and their solutions, Poynting's theorem, Time harmonic fields, Electromagnetic Spectrum.

                                              TOTAL (L:45+T:15): 60 PERIODS

OUTCOMES:
Upon completion of the course, the students would be able to
- Analyze field potentials due to static changes and static magnetic fields.
- Explain how materials affect electric and magnetic fields.
- Analyze the relation between the fields under time varying situations.
- Discuss the principles of propagation of uniform plane waves.

UNIT-1 STATIC ELECTRIC FIELD

INTRODUCTION: Electromagnetic theory is a discipline concerned with the study of charges at rest and in motion. Electromagnetic principles are fundamental to the study of electrical engineering and physics. Electromagnetic theory is also indispensable to the understanding, analysis and design of various electrical, electromechanical and electronic systems. Some of the branches of study where electromagnetic principles find application are:

RF communication, Microwave Engineering, Antennas, Electrical Machines, Satellite Communication, Atomic and nuclear research ,Radar Technology, Remote sensing, EMI EMC, Quantum Electronics, VLSI

1. Electromagnetic theory is a prerequisite for a wide spectrum of studies in the field of Electrical Sciences and Physics. Electromagnetic theory can be thought of as generalization of circuit theory. There are certain situations that can be handled exclusively in terms of field theory. In electromagnetic theory, the quantities involved can be categorized as **source quantities** and **field quantities**. Source of electromagnetic field is electric charges: either at rest or in motion. However an electromagnetic field may cause a redistribution of charges that in turn change the field and hence the separation of cause and effect is not always visible.

- *Sources of EMF:*
  Current carrying conductors.
  Mobile phones.
  Microwave oven.
  Computer and Television screen.
  High voltage Power lines.

- *Effects of Electromagnetic fields:*
  Plants and Animals.
  Humans.
  Electrical components.

- *Fields are classified as*
  Scalar field
  Vector field.

Electric charge is a fundamental property of matter. Charge exist only in positive or negative integral multiple of **electronic charge**, -e, e= 1.60×10^{-19} coulombs. [It may be noted here that in 1962, Murray Gell-Mann hypothesized **Quarks** as the basic building blocks of matters. Quarks were predicted to carry a fraction ofelectronic charge and the existence of Quarks have been experimentally verified.] Principle of conservation of charge states that the total charge (algebraic sum of positive and negative charges) of an isolated system remains unchanged, though the charges may redistribute under the influence of electric field. Kirchhoff's Current Law (KCL) is an assertion of the conservative property of charges under the implicit assumption that there is no accumulation of charge at the junction.

Electromagnetic theory deals directly with the electric and magnetic field vectors where as circuit theory deals with the voltages and currents. Voltages and currents are integrated effects of electric and magnetic fields respectively. Electromagnetic field problems involve three space variables along with the time variable and hence the solution tends to become correspondingly complex. Vector analysis is a mathematical tool with which electromagnetic concepts are more conveniently expressed and best comprehended. Since use of vector analysis in the study of electromagnetic field theory results in real economy of time and thought, we first introduce the concept of vector analysis.

Vector Analysis:

The quantities that we deal in electromagnetic theory may be either **scalar** or **vectors** [There are other class of physical quantities called **Tensors**: where magnitude and direction vary with co ordinate axes]. Scalars are quantities characterized by magnitude only and algebraic sign. A quantity that has direction as well as magnitude is called a vector. Both scalar and vector quantities are function of *time* and *position* . A field is a function that specifies a particular quantity everywhere in a region. Depending upon the nature of the quantity under consideration, the field may be a vector or a scalar field. Example of scalar field is the electric potential in a region while electric or magnetic fields at any point is the example of vector field.

A vector $\vec{A}$ can be written as, $\vec{A} = \hat{a}\, A$, where, $A = |\vec{A}|$ is the magnitude and $\hat{a} = \dfrac{\vec{A}}{|A|}$ is the unit vector which has unit magnitude and same direction as that of $\vec{A}$.

Two vector $\vec{A}$ and $\vec{B}$ are added together to give another vector $\vec{C}$. We have

$$\vec{C} = \vec{A} + \vec{B} \quad \text{...............(1.1)}$$

Let us see the animations in the next pages for the addition of two vectors, which has two rules: **1: Parallelogram law** and **2: Head & tail rule**
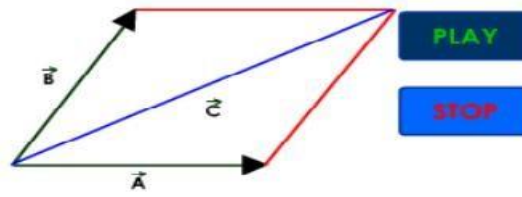
PARALLELOGRAM RULE FOR VECTOR ADDITION

USE THE PLAY AND STOP BUTTONS TO VIEW HOW THE
VECTORS A AND B ARE ADDED AND THE RESULTANT C IS
PRODUCED

Fig 1.1(a):Vector Addition(Parallelogram Rule)
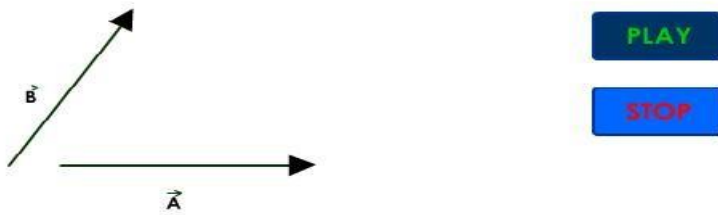
HEAD TO TAIL RULE FOR VECTOR ADDITION

USE THE PLAY AND STOP BUTTONS TO VIEW HOW THE
VECTORS A AND B ARE ADDED AND THE RESULTANT C IS
PRODUCED

Fig 1.1(b): Vector Addition (Head & Tail Rule)

Vector Subtraction is similarly carried out: $\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ ....(1.2)
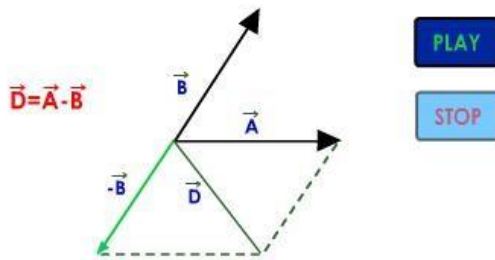
**CLICK PLAY AND STOP TO SEE THE VECTOR SUBTRATION
OF A AND B**

Fig 1.2: Vector subtraction

Scaling of a vector is defined as $\vec{C} = \alpha\vec{B}$, where $\vec{C}$ is scaled version of vector $\vec{B}$ and α is a scalar.

Some important laws of vector algebra are:

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$ Commutative Law .... (1.3)

3

$$\vec{A} + \left(\vec{B} + \vec{C} \right) = \left(\vec{A} + \vec{B} \right) + \vec{C}$$

Associative Law ....(1.4)

$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B}$$

Distributive Law ....(1.5)

The position vector $\vec{r_Q}$ of a point P is the directed distance from the origin (O) to P, i.e., $\vec{r_Q} = \overrightarrow{OP}$.
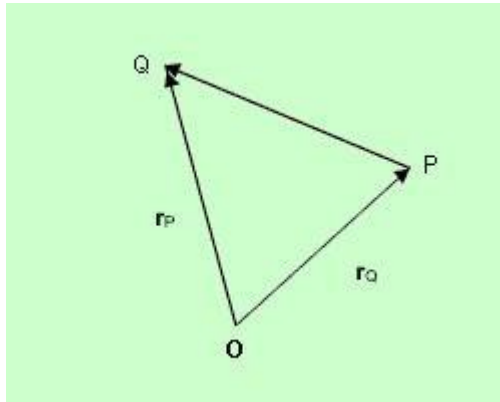
Fig 1.3: Distance Vector

If $\vec{r_Q} = $ OP and $\vec{r_P} = $ OQ are the position vectors of the points P and Q then the distance vector

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r_P} - \vec{r_Q}$$

Product of Vectors

When two vectors $\vec{A}$ and $\vec{B}$ are multiplied, the result is either a scalar or a vector depending how the two vectors were multiplied. The two types of vector multiplication are:

Scalar product (or dot product) $\vec{A} \cdot \vec{B}$ gives a scalar.

Vector product (or cross product) $\vec{A} \times \vec{B}$ gives a vector.

The dot product between two vectors is defined as $\vec{A} \cdot \vec{B} = |A||B|cos\theta AB$ ....(1.6)

Vector product $\vec{A} \times \vec{B} = |A||B|\sin \theta_{AB} \cdot \vec{n}$

$\vec{n}$ is unit vector perpendicular to $\vec{A}$ and $\vec{B}$
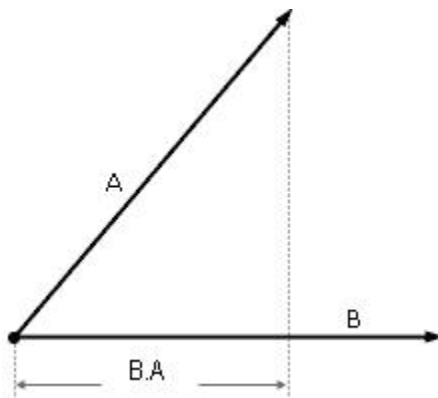
Fig 1.4: Vector dot product

The dot product is commutative i.e., $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ and distributive i.e., $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$. Associative law does not apply to scalar product.

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$. $\vec{A} \times \vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, the magnitude is given by $|A||B|\sin\theta_{AB}$ and direction is given by right hand rule as explained in Figure 1.5.
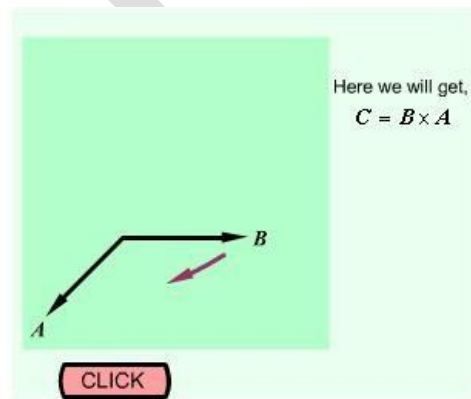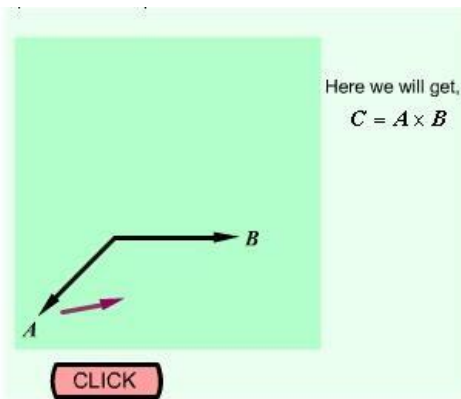
Fig 1.5 :Illustrating the left thumb rule for determining the vector cross product
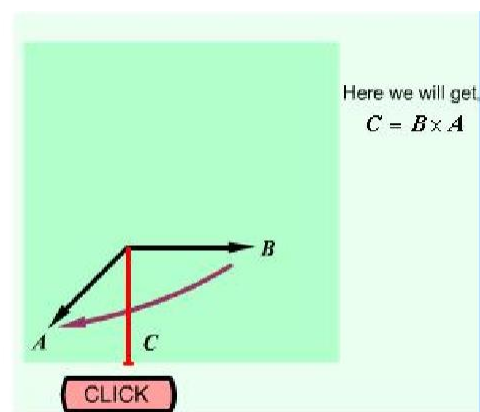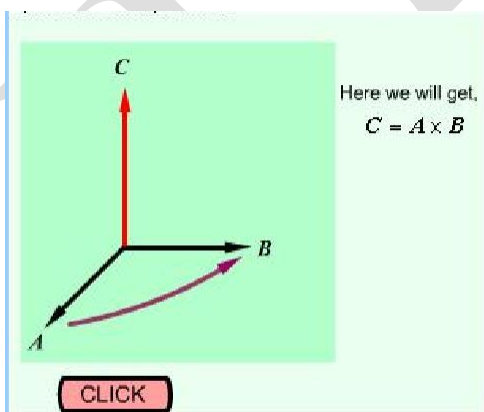
Fig 1.5 :Illustrating the left thumb rule for determining the vector cross product

$$\vec{A} \times \vec{B} = \hat{a}_n \, AB \sin \theta_{AB} \dots \dots \dots \dots \dots (1.7)$$

where $\hat{a}_n$ is the unit vector given by, $\hat{a}_n = \dfrac{\vec{A} \times \vec{B}}{\left| \vec{A} \times \vec{B} \right|}$. The

following relations hold for vector product.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \qquad \text{i.e., cross product is non commutative} \dots\dots (1.8)$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad \text{i.e., cross product is distributive} \dots\dots (1.9)$$

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \qquad \text{i.e., cross product is non associative} \dots\dots (1.10)$$

Scalar and vector triple product :

Scalar triple product $\dots \dots \dots \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \dots \dots \dots (1.11)$

Vector triple product $\dots \dots \dots \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \dots \dots \dots (1.12)$

Co-ordinate Systems

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in a **curvilinear** coordinate system that may be **orthogonal** or **non-orthogonal** .

An orthogonal system is one in which the co-ordinates are mutually perpendicular. Non- orthogonal co-ordinate systems are also possible, but their usage is very limited in practice .

Let u = constant, v = constant and w = constant represent surfaces in a coordinate system,

the surfaces may be curved surfaces in general. Furthur, let $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$ be the unit vectors in the three coordinate directions(base vectors). In a general right handed orthogonal curvilinear systems, the vectors satisfy the following relations :

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w$$

$$\hat{a}_v \times \hat{a}_w = \hat{a}_u$$

$$\hat{a}_w \times \hat{a}_u = \hat{a}_v \quad ....(1.13)$$

These equations are not independent and specification of one will automatically imply the other two. Furthermore, the following relations hold

$$\hat{a}_u \cdot \hat{a}_v = \hat{a}_v \cdot \hat{a}_w = \hat{a}_w \cdot \hat{a}_u = 0$$

$$\hat{a}_u \cdot \hat{a}_u = \hat{a}_v \cdot \hat{a}_v = \hat{a}_w \cdot \hat{a}_w = 1 \quad ....(1.14)$$

A vector can be represented as sum of its orthogonal components, $\vec{A} = A_u \hat{a}_u + A_v \hat{a}_v + A_w \hat{a}_w$ ....(1.15)

In general u, v and w may not represent length. We multiply u, v and w by conversion factors $h1, h2$ and $h3$ respectively to convert differential changes du, dv and dw to corresponding changes in length d$l1$, d$l2$, and d$l3$. Therefore

$$d\vec{l} = \hat{a}_u \, dl_1 + \hat{a}_v \, dl_2 + \hat{a}_w \, dl_3$$

$$= h_1 du \, \hat{a}_u + h_2 dv \, \hat{a}_v + h_3 dw \, \hat{a}_w \quad ....(1.16)$$

In the same manner, differential volume dv can be written as $dv = h_1 h_2 h_3 \, du\, dv\, dw$ and

differential area d$s1$ normal to $\hat{a}_u$ is given by, $ds_1 = h_2 h_3 \, dv\, dw$. In the same manner,

differential areas normal to unit vectors $\hat{a}_v$ and $\hat{a}_w$ can be defined.

In the following sections we discuss three most commonly used orthogonal co-ordinate systems, viz:

> 1. **Cartesian (or rectangular) co-ordinate system**
> 2. **Cylindrical co-ordinate system**
> 3. **Spherical polar co-ordinate system**

Cartesian Co-ordinate System :

In Cartesian co-ordinate system, we have, $(u,v,w) = (x,y,z)$. A point $P(x0, y0, z0)$ in Cartesian co-ordinate system is represented as intersection of three planes $x = x0$, $y = y0$ and $z = z0$. The unit vectors satisfies the following relation:

Cartesian Coordinate System
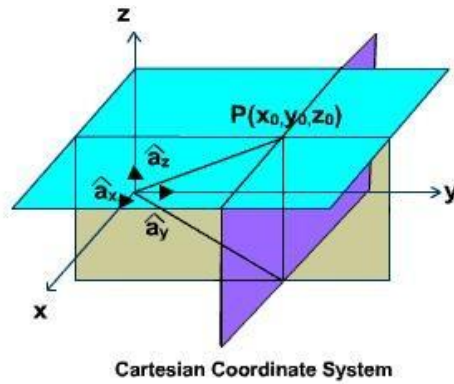
Fig 1.6 Intersection of three planes

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$

$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$

$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_x . \hat{a}_y = \hat{a}_y . \hat{a}_z = \hat{a}_z . \hat{a}_x = 0$$

$$\hat{a}_x . \hat{a}_x = \hat{a}_y . \hat{a}_y = \hat{a}_z . \hat{a}_z = 1$$

$$\overrightarrow{OP} = \hat{a}_x \, x_0 + \hat{a}_y \, y_0 + \hat{a}_z \, z_0$$

In cartesian co-ordinate system, a vector $\vec{A}$ can be written as $\vec{A} = \hat{a}_x \, A_x + \hat{a}_y \, A_y + \hat{a}_z \, A_z$.

The dot and cross product of two vectors $\vec{A}$ and $\vec{B}$ can be written as follows:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad ....(1.19)$$

$$\vec{A} \times \vec{B} = \hat{a}_x (A_y B_z - A_z B_y) + \hat{a}_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$....................(1.20)$$

Since x, y and z all represent lengths, $h_1 = h_2 = h_3 = 1$. The differential length, area

$$d\vec{l} = dx \, a_x + dy \, a_y + dz \, a_z$$

and volume are defined respectively as

$$d\vec{s}_x = dydz \, a_x$$

$$d\vec{s}_y = dxdz \, \hat{a}_y \qquad ....(1.21)$$

$$d\vec{s}_z = dxdy \, \hat{a}_z$$

$$dv = dxdydz \qquad ....(1.22)$$

Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u,v,w)=(r,\phi,z)$ a point $P(r_0,\phi_0,z_0)$ is determined as the point of intersection of a cylindrical surface $r = r_0,$ half plane

containing the z-axis and making an angle $\phi=\phi_0$; with the xz plane and a plane parallel to *xy* plane located at *z=z0* as shown in figure.

In cylindrical coordinate system, the unit vectors satisfy the following relations

A vector $\vec{A}$ can be written as , $\vec{A}=A_\rho \hat{a}_\rho+A_\phi \hat{a}_\phi+A_z \hat{a}_z$ .... (1.24)

The differential length is defined as,

$$d\vec{l}=\hat{a}_\rho d\rho+\rho d\phi\hat{a}_\phi+dz\,\hat{a}_z \qquad h_1=1, h_2=\rho, h_3=1 \quad ....(1.25)$$

$$\hat{a}_\rho\times\hat{a}_\phi=\hat{a}_z$$
$$\hat{a}_\phi\times\hat{a}_z=\hat{a}_\rho$$
$$\hat{a}_z\times\hat{a}_\rho=\hat{a}_\phi \quad ....(1.23)$$
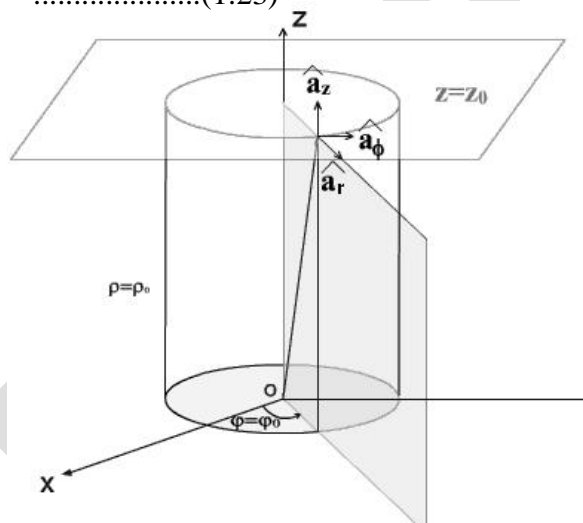
Fig 1.7 cylindrical co-ordinate system
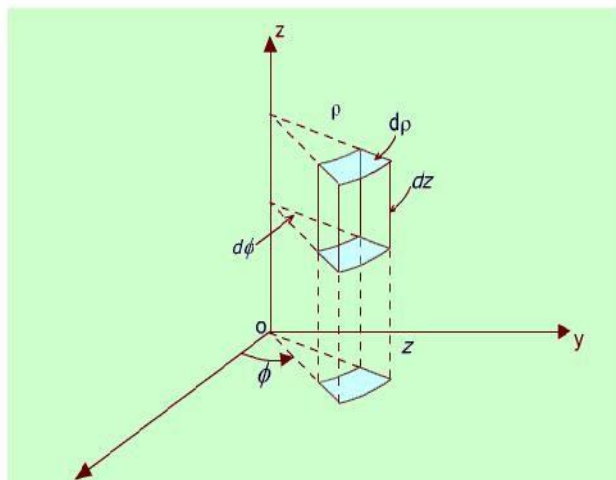
Differential areas are:

$$\vec{ds}_\rho=\rho d\phi dz\,\hat{a}_\rho$$
$$\vec{ds}_\phi=d\rho dz\,\hat{a}_\phi \quad ....(1.26)$$
$$\vec{ds}_z=\rho d\phi d\rho\,\hat{a}_z$$

Differential volume,

$$d\upsilon=\rho d\rho d\phi dz \quad ....(1.27)$$

Fig1.8 cylindrical system surface

Transformation between Cartesian and Cylindrical coordinates:

Let us consider $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ is to be expressed in Cartesian co-ordinate as

$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$. In doing so we note that $A_x = \vec{A}.\hat{a}_x = \left(\hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z \right).\hat{a}_x$

and it applies for other components as well.
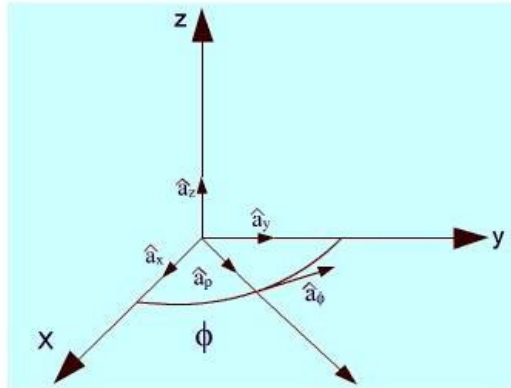
$$\hat{a}_\rho.\hat{a}_x = \cos\phi$$
$$\hat{a}_\rho.\hat{a}_y = \sin\phi$$
$$\hat{a}_\phi.\hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi \qquad \text{............(1.28)}$$
$$\hat{a}_\phi.\hat{a}_y = \cos\phi$$

Therefore we can write,

$$A_x = \vec{A}.\hat{a}_x = A_\rho \cos\phi - A_\phi \sin\phi$$
$$A_y = \vec{A}.\hat{a}_y = A_\rho \sin\phi + A_\phi \cos\phi \qquad \text{..........(1.29)}$$
$$A_z = \vec{A}.\hat{a}_z = A_z$$

Fig 1.9 : Unit Vectors in Cartesian and Cylindrical Coordinates

These relations can be put conveniently in the matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_y \\ A_z \end{bmatrix} \qquad \text{...................(1.30)}$$

A_ρ, A_ϕ and A_z themselves may be functions of ρ, ϕ and z as:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z \qquad \text{...........................(1.31)}$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

The inverse relationships are: ....$z = z$.................................... (1.32)
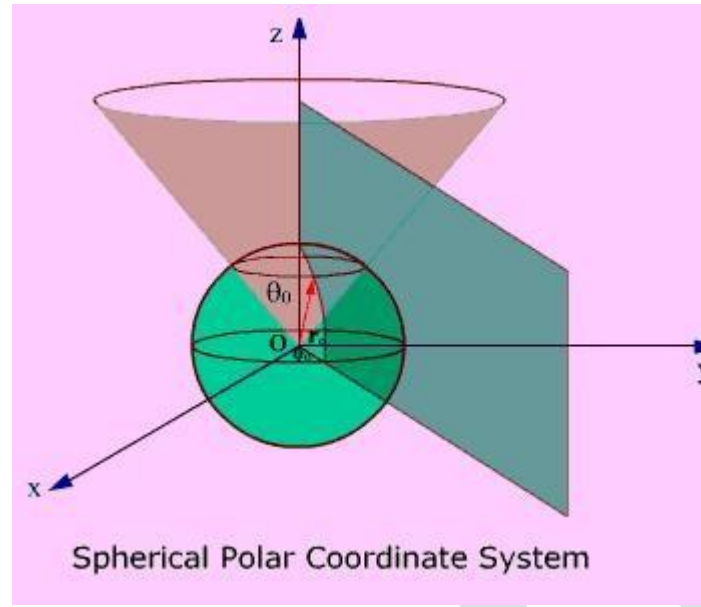
Fig 1.10: Spherical Polar Coordinate System

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: Finding the component vectors and then variable transformation.

Spherical Polar Coordinates:

For spherical polar coordinate system, we have, $(u,v,w) = (r,\theta,\phi)$. A point $P(r_0,\theta_0,\phi_0)$ is represented as the intersection of

(i) Spherical surface *r=r0*

(ii) Conical surface $\theta = \theta_0$,and

(iii) half plane containing z-axis making angle $\phi = \phi_0$ with the *xz* plane as shown in the figure 1.10.

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$

$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$

$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$

The unit vectors satisfy the following relationships: .... (1.33)

The orientation of the unit vectors are shown in the figure 1.11.
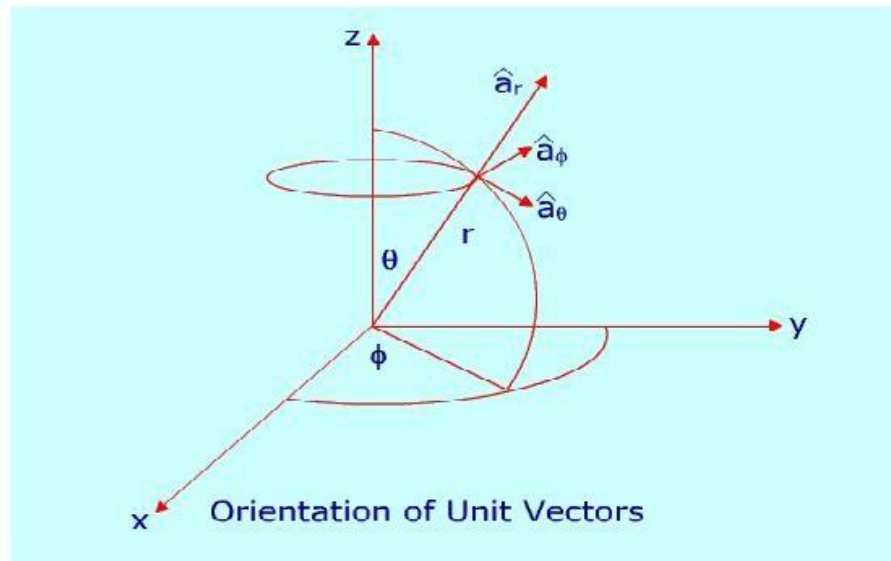
Fig 1.11: Orientation of Unit Vectors

A vector in spherical polar co-ordinates is written as : $\vec{A} = A_r \, \hat{a_r} + A_\theta \, \hat{a_\theta} + A_\phi \, \hat{a_\phi}$ and

$$d\vec{l} = \hat{a_r} \, dr + \hat{a_\theta} \, rd\theta + \hat{a_\phi} \, r \sin\theta d\phi$$

For spherical polar coordinate system we have $h1=1$, $h2= r$ and $h3 = r \sin\theta$.
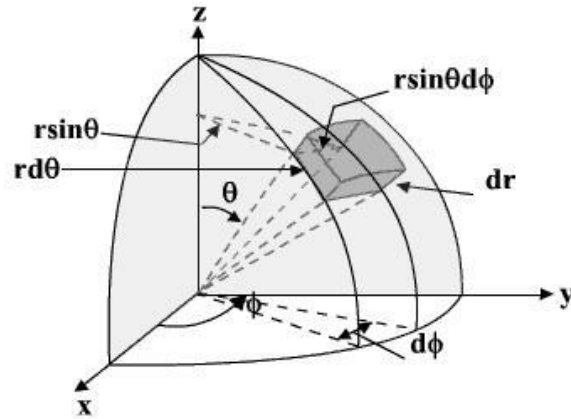
Fig 1.12(a) : Differential volume in s-p coordinates
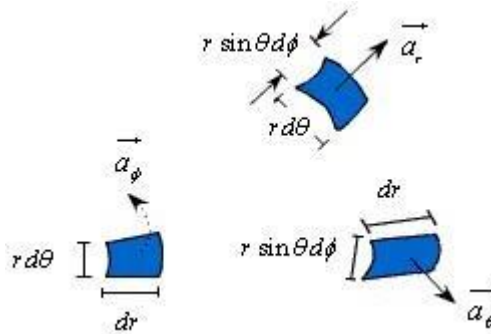
Fig 1.12(b) : Exploded view

With reference to the Figure 1.12, the elemental areas are:

$$ds_r = r^2 \sin\theta d\theta d\phi \hat{a}_r$$
$$ds_\theta = r \sin\theta dr d\phi \hat{a}_\theta$$
$$ds_\phi = r dr d\theta \hat{a}_\phi \qquad ....(1.34)$$

and elementary volume is given by

$$d\upsilon = r^2 \sin\theta dr d\theta d\phi ....(1.35)$$

Coordinate transformation between rectangular and spherical polar:

With reference to the figure 1.13 ,we can write the following equations:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi$$

$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi$$

$$\hat{a}_\theta \cdot \hat{a}_z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$$

$$\hat{a}_\phi \cdot \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_\phi \cdot \hat{a}_z = 0$$

....(1.36)
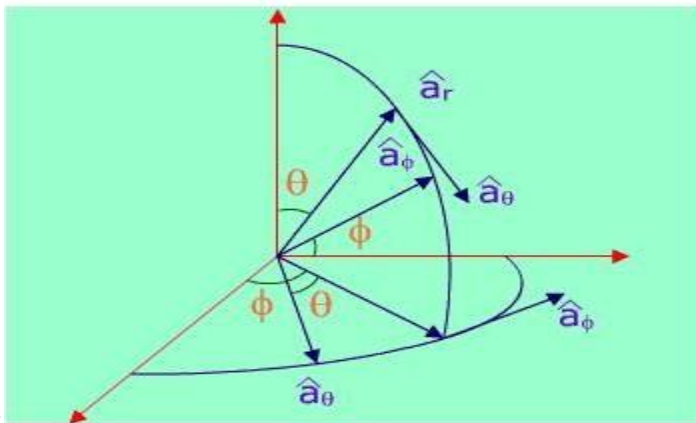
Fig 1.13: Coordinate transformation

Given a vector $\vec{A} = A_r\,\hat{a}_r + A_\theta\,\hat{a}_\theta + A_\phi\,\hat{a}_\phi$ in the spherical polar coordinate system, its component in the cartesian coordinate system can be found out as follows:

$$A_x = \vec{A}\cdot\hat{a}_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$ ....(1.37)

Similarly,

$$A_y = \vec{A}\,\hat{a}_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi\cos\phi \quad\text{.............................(1.38a)}$$

$$A_z = \vec{A}\,\hat{a}_z = A_r \cos\theta - A_\theta \sin\theta \quad\text{..................................(1.38b)}$$

The above equation can be put in a compact form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \quad\text{..............................(1.39)}$$

The components A_r, A_θ and A_ϕ themselves will be functions of r, θ and ϕ. r, θ and ϕ are related to x,y and z as:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta \quad\text{...................(1.40)}$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2} \quad\text{......................................(1.41a)}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad\text{...............................(1.41b)}$$

$$\phi = \tan^{-1}\frac{y}{x} \quad\text{...(1.41c)}$$

Using the variable transformation listed above, the vector components, which are functions of variables of one coordinate system, can be transformed to functions of variables of other coordinate system and a total transformation can be done.

Line, surface and volume integrals

In electromagnetic theory, we come across integrals, which contain vector functions. Some representative integrals are listed below:

$$\int_V \vec{F}\,dv \qquad \int_C \phi\,\vec{dl} \qquad \int_C \vec{F}.\vec{dl} \qquad \int_S \vec{F}.\vec{ds}$$

In the above integrals, $\vec{F}$ and ϕ respectively represent vector and scalar function of space coordinates. *C*,*S* and *V* represent path, surface and volume of integration. All these integrals are evaluated using extension of the usual one-dimensional integral as the limit of a sum, i.e., if a function *f(x)* is defined over arrange *a* to *b* of values of *x*, then the integral is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f_i\delta x_i$$

.................................(1.42)

where the interval (*a*,*b*) is subdivided into n continuous interval of lengths $\delta x_1, \ldots\ldots\ldots, \delta x_n$.

Line Integral: Line integral $\int_C \vec{E}\cdot\vec{dl}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.

Figure : Line Integral
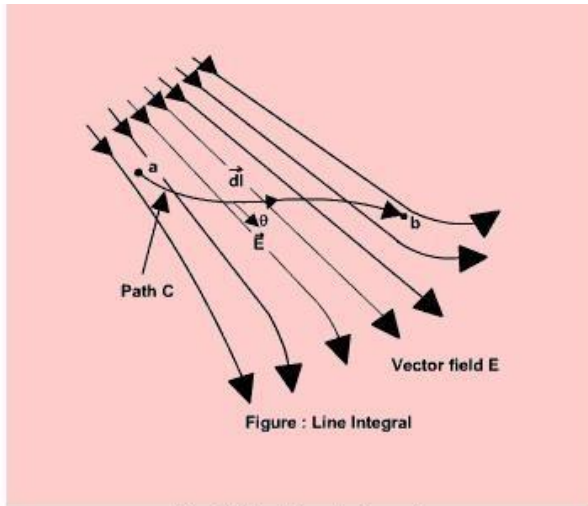
Fig 1.14: Line Integral

As shown in the figure 1.14, given a vector $\vec{E}$ around *C*, we define the integral

$$\int_C \vec{E}.\vec{dl} = \int_a^b E\cos\theta\,dl$$

as the line integral of E along the curve C.

If the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and is called the circulation of $\vec{E}$ around *C* and denoted as $\oint_C \vec{E}.\vec{dl}$ as shown in the figure 1.15.
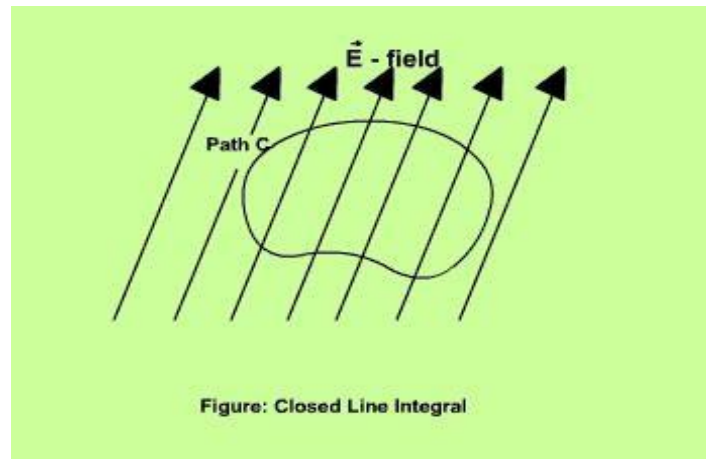
Fig 1.15: Closed Line Integral

Surface Integral :

Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or the flux of $\vec{A}$ through S as

$$\psi = \int_S A\cos\theta \, dS = \int_S \vec{A}.\hat{a}_n \, dS = \int_S \vec{A} d\vec{S}$$

as surface integral over surface S.
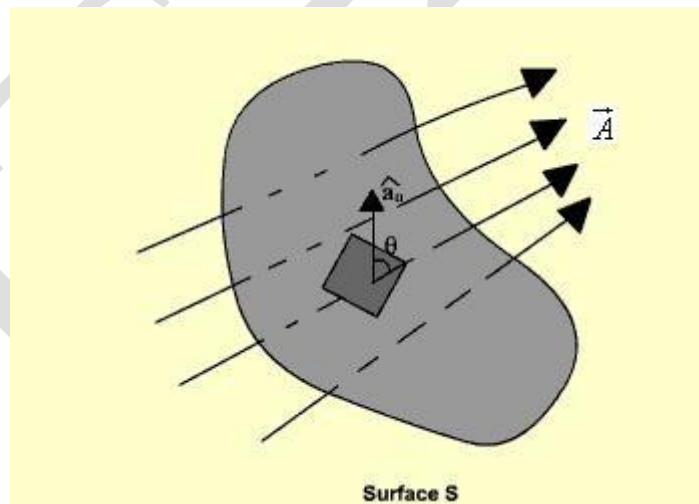
Fig 1.16 : Surface Integral

If the surface integral is carried out over a closed surface, then we write

$$\psi = \oint_S \vec{A} d\vec{S}$$

Volume Integrals:

We define $\int_V f\,dV$ or $\iiint f\,dV$ as the volume integral of the scalar function f (function of spatial coordinates) over the volume V. Evaluation of integral of the form $\int_V \vec{F}\,dV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{F}$

The Del Operator :

The vector differential operator ∇ was introduced by Sir W. R. Hamilton and later on developed by P. G. Tait.

Mathematically the vector differential operator can be written in the general form as:

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w$$ ....(1.43)

Gradient of a Scalar function:

In Cartesian coordinates:

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$ ....(1.44)

In cylindrical coordinates:

$$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$ ....(1.45)

and in spherical polar coordinates:

$$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$ ....(1.46)

Let us consider a scalar field $V(u,v,w)$, a function of space coordinates.

Gradient of the scalar field V is a vector that represents both the magnitude and direction of the maximum space rate of increase of this scalar field V.
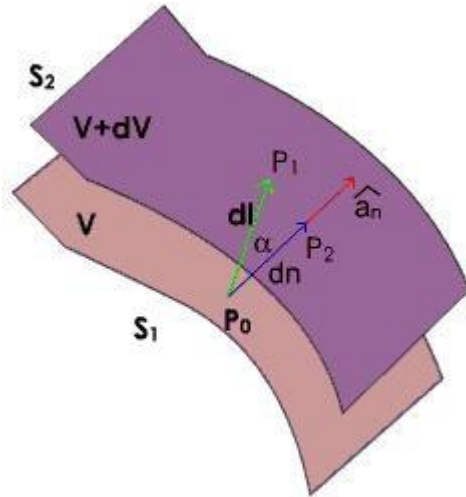
Fig 1.17 : Gradient of a scalar function

As shown in figure 1.17, let us consider two surfaces S_1 and S_2 where the function V has constant magnitude and the magnitude differs by a small amount dV. Now as one moves from S_1 to S_2, the magnitude of spatial rate of change of V i.e. dV/dl depends on the direction of elementary path length dl, the maximum occurs when one traverses from S_1 to S_2 along a path normal to the surfaces as in this case the distance is minimum.

By our definition of gradient we can write:

$$\text{grad}V = \frac{dV}{dn}\hat{a}_n = \nabla V$$ ....(1.47)

since $\vec{dn}$ which represents the distance along the normal is the shortest distance between the two surfaces.

For a general curvilinear coordinate system

$$\vec{dl} = \hat{a}_u\, dl_u + \hat{a}_v\, dl_v + \hat{a}_w\, dl_w = \left(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w \right)$$ ....(1.48)

Further we can write

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \nabla V \cdot \hat{a}_l$$ ....(1.49)

Hence,

$$dV = \nabla V.dl = \nabla V.(h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w)$$ ....(1.50)

Also we can write,
$$dV = \frac{\partial V}{\partial l_u} dl_u + \frac{\partial V}{\partial l_v} dl_v + \frac{\partial V}{\partial l_w} dl_w$$

$$= \left(\frac{\partial V}{\partial l_u} \hat{a}_u + \frac{\partial V}{\partial l_v} \hat{a}_v + \frac{\partial V}{\partial l_w} \hat{a}_w \right) \cdot (dl_u \hat{a}_u + dl_v \hat{a}_v + dl_w \hat{a}_w)$$

$$= \left(\frac{\partial V}{h_1 \partial u} \hat{a}_u + \frac{\partial V}{h_2 \partial v} \hat{a}_v + \frac{\partial V}{h_3 \partial w} \hat{a}_w \right) \cdot (h_1 du \hat{a}_u + h_2 dv \hat{a}_v + h_3 dw \hat{a}_w)$$

...........................(1.51)

By comparison we can write,

$$\nabla V = \frac{1}{h_1} \frac{\partial V}{\partial u} \hat{a}_u + \frac{1}{h_2} \frac{\partial V}{\partial v} \hat{a}_v + \frac{1}{h_3} \frac{\partial V}{\partial w} \hat{a}_w$$

....(1.52)

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written as:
In Cartesian coordinates:

$$\nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$$

....(1.53)

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$

....(1.54)

and in spherical polar coordinates:

$$\nabla V = \frac{\partial V}{\partial r} \hat{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi$$

....(1.55)

The following relationships hold for gradient operator.

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = V \nabla U + U \nabla V$$
$$\nabla(\frac{U}{V}) = \frac{V \nabla U - U \nabla V}{V^2}$$
$$\nabla V^n = n V^{n-1} \nabla V$$

....(1.56)

where *U* and *V* are scalar functions and *n* is an integer.

It may further be noted that since magnitude of $\dfrac{dV}{dl} (= \Delta V . \hat{a_1})$ depends on the direction of d*l*, it is called the **directional derivative**. If $A = \Delta V,$ V is called the scalar potential function of the vector function $\vec{A}$.

Divergence of a Vector Field:

In study of vector fields, directed line segments, also called flux lines or streamlines, represent field variations graphically. The intensity of the field is proportional to the density of lines. For example, the number of flux lines passing through a unit surface *S* normal to the vector measures the vector field strength.
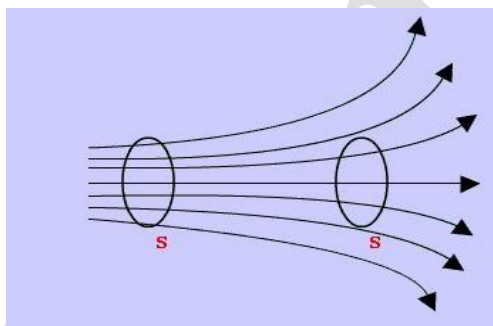
Fig 1.18: Flux Lines

We have already defined flux of a vector field as

$$\psi = \int_S A \cos \theta ds = \int_S \vec{A} \cdot \hat{a_n} ds = \int_S \vec{A} \cdot d\vec{s}$$

....(1.57)

For a volume enclosed by a surface,

$$\psi = \oint_S \vec{A} \cdot d\vec{s}$$

....(1.58)

We define the divergence of a vector field $\vec{A}$ at a point *P* as the net outward flux from a volume enclosing *P*, as the volume shrinks to zero.

$$div \ \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \dfrac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta v}$$

....(1.59)

Here ΔV is the volume that encloses *P* and *S* is the corresponding closed surface.
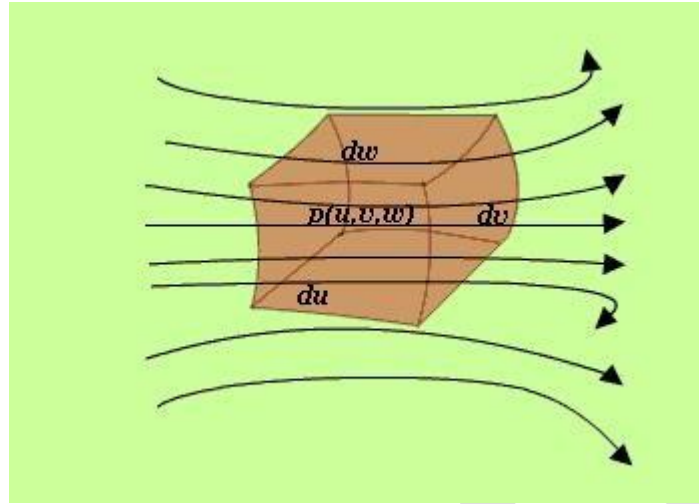
Fig 1.19: Evaluation of divergence in curvilinear coordinate

Let us consider a differential volume centered on point P(u,v,w) in a vector field $\vec{A}$. The flux through an elementary area normal to u is given by ,

$$\phi_u = \vec{A} \cdot \hat{a}_u h_2 h_3 \, dvdw \quad ....$$
$$................(1.60)$$

Net outward flux along u can be calculated considering the two elementary surfaces perpendicular to u .

$$\left[h_2 h_3 A_u \Big|_{\left(u+\frac{du}{2},v,w\right)} - h_2 h_3 A_u \Big|_{\left(u-\frac{du}{2},v,w\right)} \right] dvdw \cong \frac{\partial \left(h_2 h_3 A_u \right)}{\partial u} dudvdw$$

$$........................$$

...............(1.61) Considering the contribution from all six surfaces that enclose the volume, we can write

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A} \cdot \vec{ds}}{\Delta v} = \frac{dudvdw \dfrac{\partial \left(h_2 h_3 A_u \right)}{\partial u} + dudvdw \dfrac{\partial \left(h_1 h_3 A_v \right)}{\partial v} + dudvdw \dfrac{\partial \left(h_1 h_2 A_w \right)}{\partial w}}{h_1 h_2 h_3 \, dudvdw}$$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial \left(h_2 h_3 A_u \right)}{\partial u} + \frac{\partial \left(h_1 h_3 A_v \right)}{\partial v} + \frac{\partial \left(h_1 h_2 A_w \right)}{\partial w} \right]$$
$$..(1.62)$$

Hence for the Cartesian, cylindrical and spherical polar coordinate system, the expressions for divergence ca written as:

In Cartesian coordinates:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ ....(1.63)

In cylindrical coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial \left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$ ....(1.64)

and in spherical polar coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial \left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin \theta}\frac{\partial \left(\sin \theta A_\theta\right)}{\partial \theta} + \frac{1}{r \sin \theta}\frac{\partial A_\phi}{\partial \phi}$$ ....(1.65)

In connection with the divergence of a vector field, the following can be noted

- Divergence of a vector field gives a scalar.

$$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$$
$$\nabla \cdot (V\vec{A}) = V\nabla \cdot \vec{A} + \vec{A} \cdot \nabla V$$ ....(1.66)

Divergence theorem :
Divergence theorem states that the volume integral of the divergence of vector field is equal to the net outward flux of the vector through the closed surface that bounds the

volume. Mathematically, $\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot d\vec{s}$

Proof:

Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large

number of cells. Let the kth cell has a volume ΔV_k and the corresponding surface is denoted by S_k. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells are considered, we actually get the net outward flux through the surface surrounding the volume. Hence we can write:

$$\oint_S \vec{A} \cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A} \cdot d\vec{s} = \sum_k \frac{\oint_{S_k} \vec{A} \cdot d\vec{s}}{\Delta V_k} \Delta V_k$$ ....(1.67)

In the limit, that is when $K \to \infty$ and $\Delta V_k \to 0$ the right hand of the expression can be written as $\int_V \nabla.A dV$.

Hence we get $\oint_S \vec{A} \cdot d\vec{S} = \int_V \nabla \cdot A dV$, which is the divergence theorem.

Curl of a vector field:

We have defined the circulation of a vector field *A* around a closed path as $\oint_l \vec{A} \cdot d\vec{l}$.

Curl of a vector field is a measure of the vector field's tendency to rotate about a point. Curl $\vec{A}$, also written as $\nabla \times \vec{A}$ is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum.

Therefore, we can write:

$$Curl \ \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \frac{\hat{a}_n}{\Delta S} \left[\oint_l \vec{A} \cdot dl \right]_{max}$$ ....(1.68)

To derive the expression for curl in generalized curvilinear coordinate system, we first compute $\nabla \times \vec{A}.\hat{a}_n$ and to do so let us consider the figure 1.20 :

24
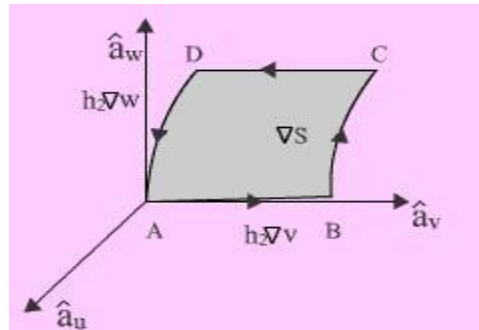
Fig 1.20: Curl of a Vector

C_1 represents the boundary of ΔS, then we can write

$$\oint_{C_1} \vec{A} \cdot \vec{dl} = \int_{AB} \vec{A} \cdot \vec{dl} + \int_{BC} \vec{A} \cdot \vec{dl} + \int_{CD} \vec{A} \cdot \vec{dl} + \int_{DA} \vec{A} \cdot \vec{dl}$$ ....(1.69)

The integrals on the RHS can be evaluated as follows:

$$\int_{AB} \vec{A} \cdot \vec{dl} = \left(A_u \hat{a}_u + A_v \hat{a}_v + A_w \hat{a}_w \right) \cdot h_2 \Delta v \hat{a}_v = A_v h_2 \Delta v$$ ....(1.70)

$$\int_{CD} \vec{A} \cdot \vec{dl} = -\left(A_v h_2 \Delta v + \frac{\partial}{\partial w}(A_v h_2 \Delta v) \Delta w \right)$$ ....(1.71)

The negative sign is because of the fact that the direction of traversal reverses. Similarly,

$$\int_{BC} \vec{A} \cdot \vec{dl} = \left(A_w h_3 \Delta w + \frac{\partial}{\partial v}(A_w h_3 \Delta w) \Delta v \right)$$ ....(1.72)

$$\int_{DA} \vec{A} \cdot \vec{dl} = -A_w h_3 \Delta w$$ ....(1.73)

Adding the contribution from all components, we can write:

$$\oint_{C_1} \vec{A} \cdot \vec{dl} = \left(\frac{\partial}{\partial v}(A_w h_3) - \frac{\partial}{\partial w}(A_v h_3) \right) \Delta v \Delta w$$ ………………………………….........(1.74)Therefore,

$$(\nabla \times \vec{A}) \cdot \hat{a}_u = \frac{\oint_{C_1} \vec{A} \cdot \vec{dl}}{h_2 h_3 \Delta v \Delta w} = \frac{1}{h_2 h_3} \left(\frac{\partial (h_3 A_w)}{\partial v} - \frac{\partial (h_2 A_v)}{\partial w} \right)$$ ....(1.75)

In the same manner if we compute for $(\nabla \times \vec{A}) \cdot \hat{a}_v$ and $(\nabla \times \vec{A}) \cdot \hat{a}_w$ we can write,

$$\nabla \times \vec{A} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 A_w)}{\partial v} - \frac{\partial(h_2 A_v)}{\partial w}\right)\hat{a}_u + \frac{1}{h_1 h_3}\left(\frac{\partial(h_1 A_u)}{\partial w} - \frac{\partial(h_3 A_w)}{\partial u}\right)\hat{a}_v + \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 A_v)}{\partial u} - \frac{\partial(h_1 A_u)}{\partial v}\right)\hat{a}_w$$

.......(1.76)

This can be written as,

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{a}_u & h_2 \hat{a}_v & h_3 \hat{a}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 A_u & h_2 A_v & h_3 A_w \end{vmatrix}$$

...............................(1.77)

In Cartesian coordinates:
$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$
....(1.78)

In Cylindrical coordinates,
$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$
.... (1.79)

In   Spherical   polar   coordinates,
$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$
.............(1.80) Curl operation exhibits the

following properties:

(i)   Curl of a vector field is another vector field.

(ii) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(iii) $\nabla \times (V\vec{A}) = \nabla V \times \vec{A} + V\nabla \times \vec{A}$

(iv) $\nabla . (\nabla \times \vec{A}) = 0$

(v) $\nabla \times \nabla V = 0$

(vi) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\nabla.\vec{B} - \vec{B}\nabla.\vec{A} + (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B}$ ....(1.81)

Stoke's theorem :

It states that the circulation of a vector field $\vec{A}$ around a closed path is equal to the integral of $\nabla \times \vec{A}$ over the surface bounded by this path. It may be noted that this equality holds provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on the surface.

i.e,

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{s} \quad ....(1.82)$$

Proof: Let us consider an area S that is subdivided into large number of cells as shown in the figure 1.21.
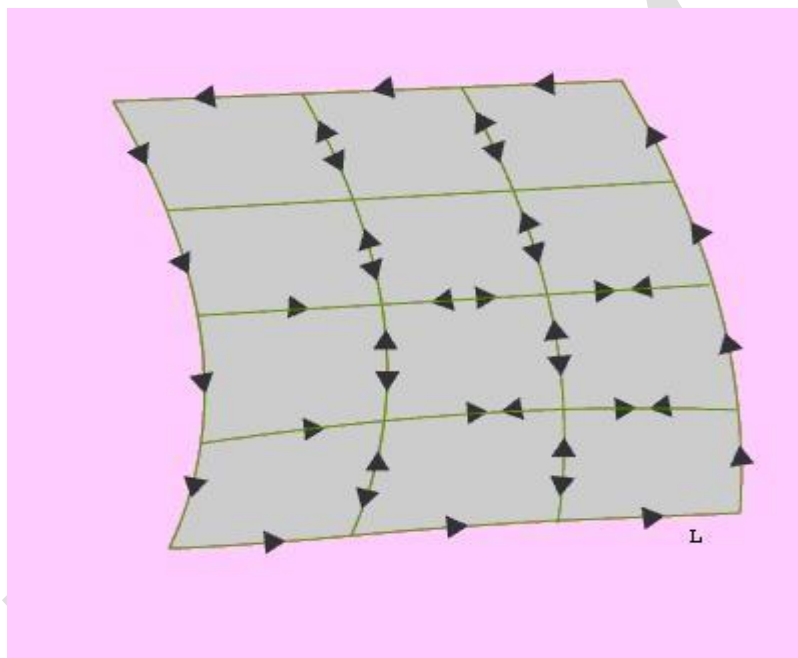
Fig 1.21: Stokes theorem

Let k^{th} cell has surface area ΔS_k and is bounded path Lk while the total area is bounded by path L. As seen from the figure that if we evaluate the sum of the line integrals around the elementary areas, there is cancellation along every interior path and we are left the line integral along path L. Therefore we can write,

$$\oint_L \vec{A} \cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A} \cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A} \cdot d\vec{l}}{\Delta S_k} \Delta S_k \quad ....(1.83)$$

As   0
                                                                $\Delta S_k \rightarrow$

$$\oint_L \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{s} \quad ....(1.84)$$

which is the stoke's theorem.

Coulomb's Law

Coulomb's Law states that the force between two point charges $Q1$ and $Q2$ is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealised model of a particle having an electric charge.

$$F = \frac{kQ_1Q_2}{R^2}$$

Mathematically, ,where k is the proportionality constant.

In SI units, $Q1$ and $Q2$ are expressed in Coulombs(C) and R is in meters.

$$k = \frac{1}{4\pi\varepsilon_0}$$

Force F is in Newtons (N) and , ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0\varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$$

Therefore..(2.1)

As shown in the Figure 2.1 let the position vectors of the point charges $Q1$ and $Q2$ are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\vec{F_{12}}$ represent the force on $Q1$ due to charge $Q2$.
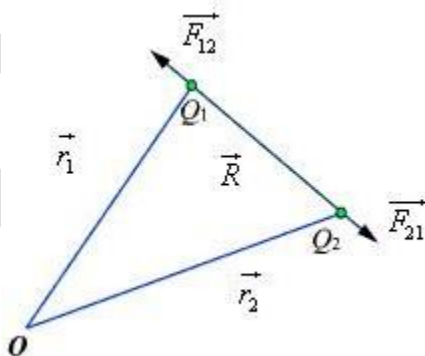
Fig 1.22: Coulomb's Law

The charges are separated by a distance of $R = |\vec{r_1} - \vec{r_2}| = |\vec{r_2} - \vec{r_1}|$. We define the unit vectors as

$$\widehat{a_{12}} = \frac{(\vec{r_2} - \vec{r_1})}{R} \quad \text{and} \quad \widehat{a_{21}} = \frac{(\vec{r_1} - \vec{r_2})}{R} \quad\quad\quad\quad\quad (2.2)$$

$$\vec{F_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r_2} - \vec{r_1})}{|\vec{r_2} - \vec{r_1}|^3}$$

$\vec{F_{12}}$ can be defined as . Similarly the force on $Q1$

due to charge $Q2$ can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, *we apply principle of superposition*. If we have N number of charges $\overrightarrow{r_1}$ $\overrightarrow{r_2}$

$Q1,Q2,.........QN$ located respectively at the points represented by the position vectors ,

$,......\overrightarrow{r_N}$,the force experienced by a charge Q located at is given by,

$$\overrightarrow{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_i(\overrightarrow{r} - \overrightarrow{r_i})}{|\overrightarrow{r} - \overrightarrow{r_i}|^3}$$

................................(2.3)

Electric Field

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\overrightarrow{E} = \lim_{Q \to 0} \frac{\overrightarrow{F}}{Q} \quad \overrightarrow{E} = \frac{\overrightarrow{F}}{Q}$$

or,... (2.4)

The electric field intensity E at a point r (observation point) due a point charge Q located at $\overrightarrow{r'}$ (source point) is given by:

$$\overrightarrow{E} = \frac{Q(\overrightarrow{r} - \overrightarrow{r'})}{4\pi\varepsilon_0 |\overrightarrow{r} - \overrightarrow{r'}|^3}$$

....*(2.5)*

For a collection of N point charges $Q1, Q2, ...QN$ located at $\overrightarrow{r_1}$, $\overrightarrow{r_2}$, , the electric field intensity at point $\overrightarrow{r'}$ is obtained as

$$\overrightarrow{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\overrightarrow{r} - \overrightarrow{r_i})}{|\overrightarrow{r} - \overrightarrow{r_i}|^3}$$

....................................(2.6)

The expression *(2.6)* can be modified suitably to compute the electric filed due to a continuous distribution of charges.
In figure 2.2 we consider a continuous volume distribution of charge d*(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\overrightarrow{r})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3} = \frac{\rho(\vec{r'})dv'(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}$$ ....*(2.7)*
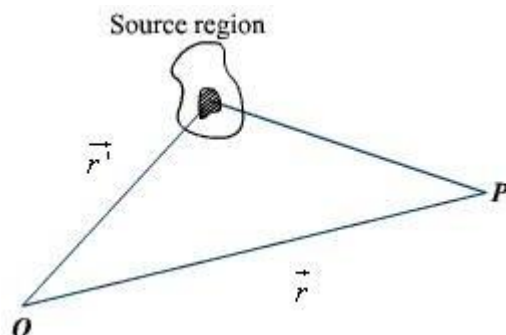

Fig 1.23: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}dv'$$ .... (2.8)

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}dl'$$ .... (2.9)

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r'})(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r'}\right|^3}ds'$$ ....(2.10)

Electric flux density:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon\vec{E}$$ ....*(2.11)*

We define the electric flux $\square$ as

$$\psi = \int_S \vec{D}.d\vec{s}$$

.....................................(2.12)

Gauss's Law: Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.
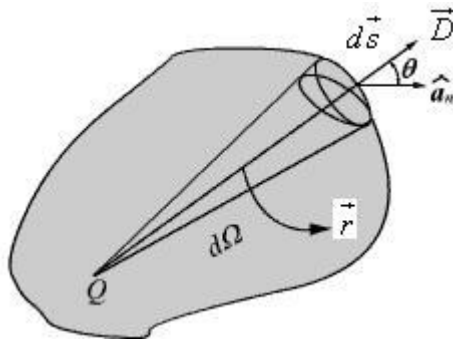

Fig 1.24: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant. The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$

....(2.13)

If we consider an elementary area ds, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2}ds\cos\theta$$

..................................(2.14)

But $\dfrac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of

Q. Therefore we can write $d\psi = \dfrac{Q}{4\pi}d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \dfrac{Q}{4\pi}\oint_S d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law. Application of Gauss's Law

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $\square$LC/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 2.4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 2.4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

..................................(2.15)

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to

the
electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write, $\rho_L l = \varepsilon_0 E.2\pi \rho l$
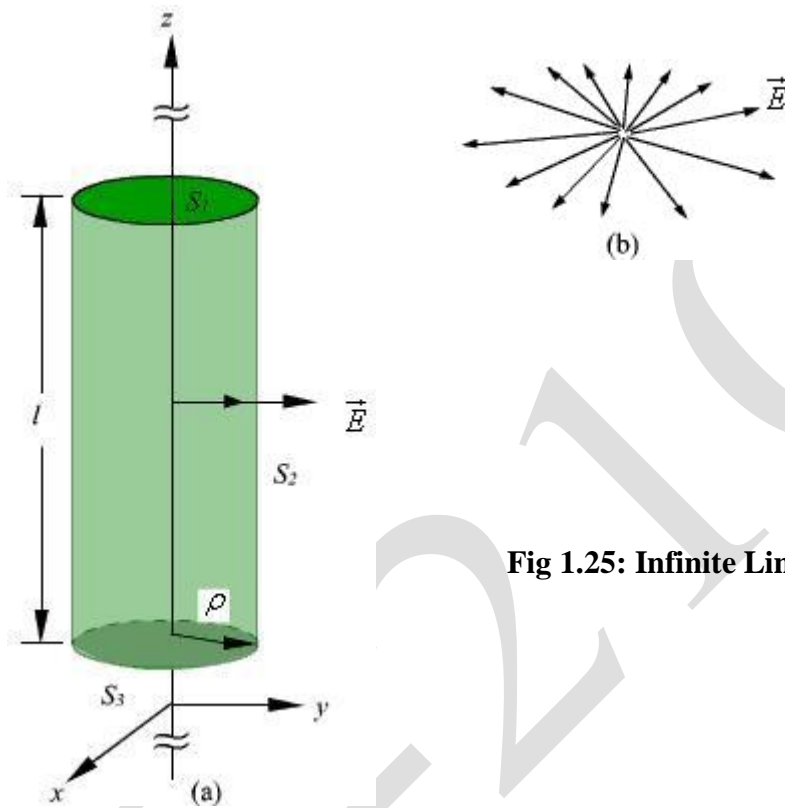
Fig 1.25: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho \quad ....(2.16)$$

Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure 2.5.

Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D} \cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y \quad ....(2.17)$$
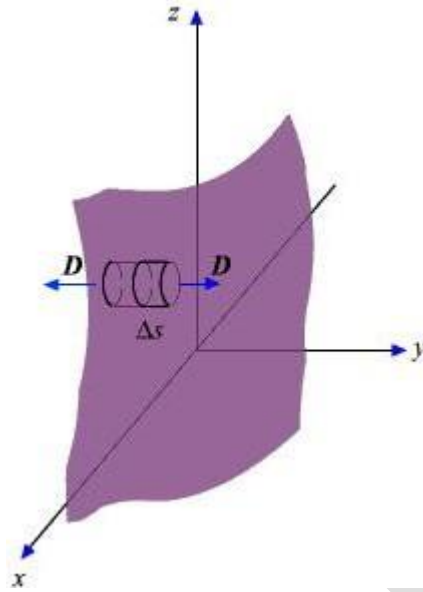
Fig 1.26: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of ρ_v C/m^3. To determine $\vec{E}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius $r < r_0$ and $r > r_0$ as shown in Fig. 2.6 (a) and Fig. 2.6(b).

For the region $r < r_0$ the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \quad \text{.........................(2.18)}$$

Fig 1.*27*:Uniformly Charged Sphere

By applying Gauss's theorem,

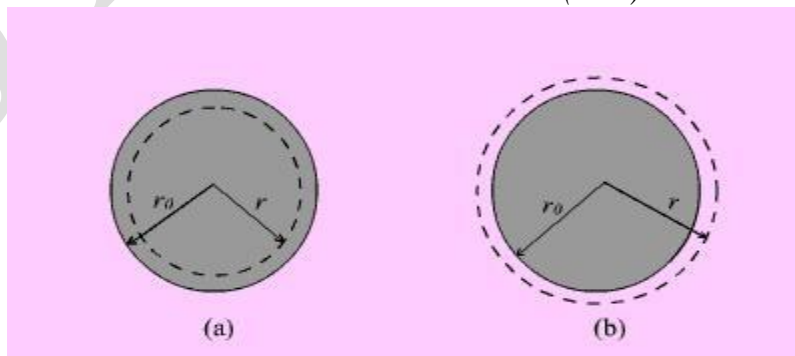$$\oint_s \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2 D_r = Q_{en}$$

.............*(2.19)*

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$

....*(2.20)*

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$

....(2.21

) By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$

....(2.22)

Gauss divergence theorem:
        The gauss law can be stated in the point form by the divergence of electric flux density is equal to the volume charge density.

Absolute Electric Potential and potential differences and its calculation.

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge Δq from a point *P* to another point *Q* as shown in the Fig. 2.8.

The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

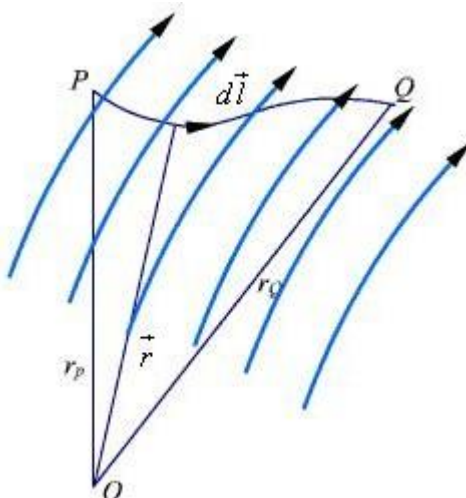$$dW = -\Delta q \vec{E}.d\vec{l}$$ ....(2.23)

Fig 1.28: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_{P}^{Q} \vec{E}\cdot d\vec{l}$$ ....(2.24)

The potential difference between two points P and Q, VPQ, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_{P}^{Q} \vec{E}\cdot d\vec{l}$$ ....(2.25)

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in

Joules/Coulomb which is referred to as **Volts**.

Let us consider a point charge Q as shown in the Fig. *2.9*.
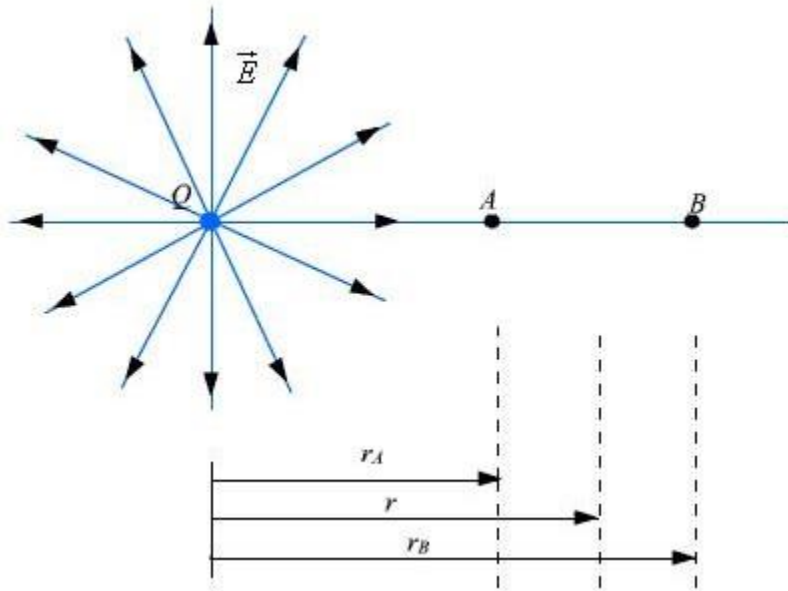
Fig 1.29: Electric Potential calculation for a point charge

Further consider the two points *A* and *B* as shown in the Fig. *2.9*. Considering the movement of a unit positive test charge from *B* to *A* , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E} \cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$

........................(2.26)

It is customary to choose the potential to be zero at infinity. Thus potential at any point ($r_A = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $r_B = 0$).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

........................(2.27)

Or, in other words,

$$V = -\int_\infty^r E.dl$$

...............................(2.28)

Let us now consider a situation where the point charge Q is not located at the origin

as shown in Fig. 2.10.

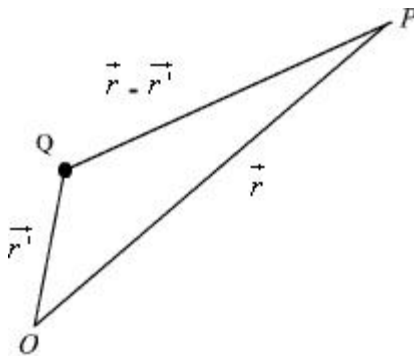**Fig *1*.30: Electric Potential due a Displaced Charge**

The potential at a point *P* becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\left|\vec{r} - \vec{r'}\right|}$$

..............................(2.29)

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also.

Let us first consider *N* point charges *Q1, Q2,.....QN* located at points with position vectors $\vec{r_1}$, $\vec{r_2}$,...... $\vec{r_N}$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r} - \vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r} - \vec{r_2}\right|} + \cdots \frac{Q_N}{\left|\vec{r} - \vec{r_N}\right|}\right)$$

..............................(2.30a)

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r} - \vec{r_n}\right|}$$

....(2.30b)

For continuous charge distribution, we replace point charges *Qn* by corresponding charge elements $\rho_l dl$ or $\rho_s ds$ or $\rho_v dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For   line   charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(\vec{r'})dl'}{|\vec{r} - \vec{r_n}|}$$

..............................(2.31)

For surface charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r'})ds'}{|\vec{r} - \vec{r_n}|}$$

....(2.32)

For volume charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r'})dv'}{|\vec{r} - \vec{r_n}|}$$

....(2.33)

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$

..............................(2.34)

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E} \cdot d\vec{l} + C$$

..............................(2.35)

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} = \frac{W}{Q}$$

.....................(2.36)

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point $P1$ to $P2$ in one path and then from point $P2$ back to $P1$ over a different path.

If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position $P1$. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $VAB = - VBA$, and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0 \quad \text{...........................(2.37)}$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E} \right) \cdot d\vec{s} = 0 \quad \text{........................(2.38)}$$

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0 \quad \text{.................................(2.39)}$$

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an irrotational

field. From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial x} dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \cdot \left(dx \hat{a}_x + dy \hat{a}_y + dz \hat{a}_z \right) = -\vec{E} \cdot d\vec{l}$$

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l} \quad \text{...........................(2.40)}$$

from which we obtain,

$$\vec{E} = -\nabla V \quad \text{.................................(2.41)}$$

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that $\vec{E}$ is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field, i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite sign and separated by a small distance.

As shown in figure 2.11, the dipole is formed by the two point charges Q and -Q

separated by a distance d , the charges being placed symmetrically about the origin. Let us consider a
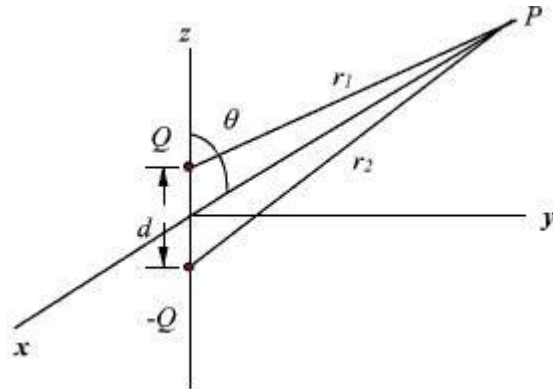point P at a distance r, where we are interested to find the field.

Fig *1.3***1 : Electric Dipole**

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \qquad (2.42)$$

When r_1 and $r_2 \gg d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and
. Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \qquad (2.43)$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r \qquad (2.44)$$

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole.

Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} \qquad (2.45)$$

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance.

If the dipole is not centered at the origin, but the dipole center lies at $\vec{r}'$, the expression for the potential can be written as:

$$V = \frac{\vec{P} \cdot (\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} \quad \text{.....................(2.46)}$$

The electric field for the dipole centered at the origin can be computed as

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right) \quad \text{.......................(2.47)}$$

$\vec{P} = Q\vec{d}$ is the magnitude of the dipole moment. Once again we note that the electric field of electric dipole varies as $1/r^3$ where as that of a point charge varies as $1/r^2$.

Electrostatic Energy and Energy Density

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges $Q1, Q2,......., QN$ are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q1 is zero. Q2 is brought in the
presence of the field of Q1, the work done $W1 = Q2V21$ where $V21$ is the potential at the location
of Q2 due to Q1. Proceeding in this manner, we can write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots\ldots\ldots + (V_{N1}Q_N + \ldots\ldots + V_{N(N-1)}Q_N)$$
$$\text{...(2.89)}$$

Had the charges been brought in the reverse order,

Therefore,

Here V_{IJ} represent voltage at the Ith charge location due to J^{th} charge. Therefore,

$$2W = V_1Q_1 + \ldots\ldots\ldots\ldots + V_NQ_N = \sum_{I=1}^{N} V_IQ_I \quad \ldots\ldots(2.91)$$

Or,

$$W = \frac{1}{2}\sum_{I=1}^{N} V_IQ_I \quad \ldots\ldots\ldots\ldots(2.92)$$

If instead of discrete charges, we now have a distribution of charges over a volume v then we can write,

$$W = \frac{1}{2}\int_v V\rho_v dv \quad \ldots\ldots\ldots\ldots(2.93)$$

where ρ_v is the volume charge density and V represents the potential function.

$$\rho_v = \nabla \cdot \vec{D} \qquad W = \frac{1}{2}\int_v \left(\nabla \cdot \vec{D}\right)V dv \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.94)$$

Using the vector identity,

$$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla \cdot \vec{D}$$, we can write

$$W = \frac{1}{2}\int_V \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V \right) dv$$

$$= \frac{1}{2}\oint_S \left(V\vec{D} \right) d\vec{s} - \frac{1}{2}\int_V \left(\vec{D}\cdot\nabla V \right) dv \qquad \text{...............(2.95)}$$

In the expression $\frac{1}{2}\oint_S \left(V\vec{D} \right) d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$,

the term $V\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r^2. Hence the integral term varies at

least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \rightarrow \infty$) the integral term tends to zero.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_V \left(\vec{D}\nabla V \right) dv = \frac{1}{2}\int_V \left(\vec{D}.\vec{E} \right) dv = \frac{1}{2}\int_V \left(\varepsilon E^2 \right) dv = \int_V w_e dv \qquad \text{...............(2.96)}$$

$w_e = \frac{1}{2}\varepsilon E^2$, is called the energy density in the electrostatic field.