Cauchy's Integral Theorem :

If $f(z)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C, then $\int_C f(z)dz = 0$

Problems :

1) Evaluate $\int_C \frac{dz}{z+4}$, where C is the circle $|z| = 2$.

Soln :

$z+4 = 0$

$z = -4$

C is $|z| = 2$

$z = -4 \Rightarrow |z| = |-4| = 4 > 2$

$\therefore z = -4$ lies outside C.

By Cauchy's Integral theorem, $\int_C \frac{dz}{z+4} = 0$.

2) Evaluate $\int_C \frac{dz}{2z-3}$, where C is the circle $|z| = 1$.

Soln :

$2z-3 = 0 \Rightarrow z = 3/2$

C is $|z| = 1$

$z = 3/2 \Rightarrow |z| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$

$z = \frac{3}{2}$ lies outside C.

By Cauchy's Integral theorem, $\int_C \frac{dz}{2z-3} = 0$.

3) Evaluate $\int_C e^z dz$ where C is $|z| = 1$.

Soln :

$f(z) = e^z$

$\therefore f(z)$ lies inside C.

$\therefore \int_C e^z dz = 0$ [By C.I.T] closed contour c, taken in the positive sense. If a is any point interior to c, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z-a}$$

(or).

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

Note :

$$\int_c \frac{f(z)}{z-a}\,dz = \begin{cases} 2\pi i\, f(a), & a \text{ lies inside } c \\ 0, & a \text{ lies outside } c \end{cases}$$

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \begin{cases} \frac{2\pi i}{n!} f^{n}(a), & a \text{ lies inside } c \\ 0, & a \text{ lies outside } c \end{cases}$$

Problems :-

1) Evaluate $\int_c \frac{z}{z-2}\,dz$ where c is $|z| = 1$ & $|z| = 3$.

Soln :

Given $\int_c \frac{z}{z-2}\,dz$

$z-2 = 0 \Rightarrow z = 2$

$\therefore$ z lies outside c, $|z| = 2$

$$\therefore \int_c \frac{z}{z-2}\,dz = 0.$$

z lies outside c, $|z| = 3$.

$$\int_c \frac{z}{z-2}\,dz = 2\pi i\, f(2)$$
$$= 2\pi i\,(2) = 4\pi i$$

2) Evaluate $\int_c \frac{z+1}{z^3+2z+4}\,dz$ where c is the circle $|z+1+i| = 2$ using cauchy's integral formula.

$$\int_C \frac{z+1}{z^2+2z+4}$$

$z^2+2z+4=0$

$$z=\frac{-2\pm\sqrt{4-16}}{2}=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm 2i\sqrt{3}}{2}=-1\pm\sqrt{3}i$$

$$\int \frac{z+1}{[z-(-1+\sqrt{3}i)]\,[z-(-1-\sqrt{3}i)]}\,dz$$

$|z+1+i| = |-1+\sqrt{3}i+1+i| = |(\sqrt{3}+1)i|$.

$= 2.732 > 2$

$z=-1+\sqrt{3}i$ lies outside C.

$|z+1+i| = |-1-\sqrt{3}i+1+i| = |(1-\sqrt{3})i|$.

$= |-1.732+1| = -0.732 < 2$

$\therefore z=-1-\sqrt{3}i$ lies inside C.

$$\int_C \frac{z+1}{z^2+2z+4}\,dz = \int \frac{z+1}{[z-(-1+\sqrt{3}i)]\,[z-(-1-\sqrt{3}i)]}\,dz$$

$$= \int \frac{\dfrac{z+1}{z-(-1+\sqrt{3}i)}}{z-(-1-\sqrt{3}i)}\,dz$$

$$= 2\pi i\, f(-1-\sqrt{3}i)$$

$$= 2\pi i\left[\frac{-1-\sqrt{3}i+1}{-1-\sqrt{3}i+1-\sqrt{3}i}\right]$$

$$= 2\pi i\left[\frac{-\sqrt{3}i}{-2\sqrt{3}i}\right] = \pi i$$

3) Evaluate $\int_C \frac{e^z}{z-1}\,dz$ if C is $|z|=2$.

Soln :

Given, $\int_C \frac{e^z}{z-1}\,dz$

$$\int_C \frac{e^z}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\, e^1 = 2\pi i e.$$

4) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1+i|=2$ using cauchy's integral formula.

Soln :

Given, $\int_C \frac{z+4}{z^2+2z+5}\,dz$

$z^2+2z+5=0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2\pm 4i}{2} = -1 \pm 2i$$

$|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$

$\therefore z = -1+2i$ lies outside c.

$|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$

$\therefore z = -1-2i$ lies inside c.

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int_C \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}\,dz$$

$$= 2\pi i\, f(-1-2i)$$

$$= 2\pi i \left[\frac{-1-2i+4}{-1-2i+1-2i}\right]$$

$$= 2\pi i \left[\frac{3-2i}{-4i}\right] = -\frac{(3-2i)}{2}\pi$$

$$= \left(\frac{-3+2i}{2}\right)\pi$$

Soln :

Given, $f(z) = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

$(z-1)(z-2) = 0 \Rightarrow z = 1, 2 < 3$ lies ~~outside~~ inside C, $|z| = 3$.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1$, $\Rightarrow A = -1$

Put $z = 2$, $\Rightarrow B = 1$

$$\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = -\frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -\int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} . dz + \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} . dz$$

$$= -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i(-1) + 2\pi i = 4\pi i.$$

6) Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz$ by cauchy's integral formula, where C is the circle $|z| = 1$.

Soln :

$$\left(z - \frac{\pi}{6}\right)^3 = 0 \Rightarrow z = \frac{\pi}{6}$$

$$|z| = \left|\frac{\pi}{6}\right| = 0.5233 < 1.$$

$z = \pi/6$ lies inside C, $|z| = 1$.

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} [\sin^6 z]_{z=\pi/6}$$

$$= \frac{d}{dz}\left[6\sin^5 z \cos z\right]$$

$$= 6\left[\sin^5 z(-\sin z) + \cos z(5\sin^4 z \cos z)\right]$$

$$= 6\left[-\sin^6 z + 5\sin^4 z \cos^2 z\right]$$

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i \times 6}{2!}\left[-\sin^6 z + 5\sin^4 z \cos^2 z\right]_{z=\pi/6}$$

$$= 6\pi i\left[-\sin^6\frac{\pi}{6} + 5\sin^4\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right)\right]$$

$$= 6\pi i\left[-\frac{1}{64} + 5\times\frac{3}{4}\times\frac{1}{16}\right]$$

$$= 6\pi i\left[\frac{-1+15}{64}\right] = \frac{21\pi i}{16}$$

7) Using cauchy's integral formula evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}dz$ where c is $|z| = 3$.

Soln:

$(z+2)(z+1)^2 = 0$

$z = -2,\ z = -1$

$|z| = |-2| = 2 < 3$

$\therefore z = -2$ lies inside c, $|z| = 3$.

$|z| = |-1| = 1 < 3$

$\therefore z = -1$ lies inside c, $|z| = 3$.

$$\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz = \int_C \frac{\frac{e^z}{(z+1)^2}}{z+2}\,dz + \int_C \frac{\frac{e^z}{z+2}}{(z+1)^2}\,dz$$

$$= 2\pi i\, f(-2) + 2\pi i\, f'(-1)$$

$$= 2\pi i\,\frac{e^{-2}}{(-1)^2} + 2\pi i\left[\frac{(z+2)e^z - e^z(1)}{(z+2)^2}\right]_{z=-1}$$

$$= 2\pi i\, e^{-2} + 2\pi i\left[\frac{(1)e^{-1} - e^{-1}}{(-1+2)^2}\right]$$

8) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3/2$.

Soln :

Given, $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$

$z = 0, 1, 2$

$|z| = 0 < 3/2$

$|z| = |1| = 1 < 3/2$

$z = 0, 1$ lies inside C.

$|z| = |2| = 2 > 3/2$

$\therefore z = 2$ lies outside C.

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int \frac{\frac{4-3z}{z-2}}{z(z-1)}\,dz$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+B(z)}{z(z-1)}$$

$$1 = A(z-1) + B(z)$$

Put $z = 0 \Rightarrow A = -1$

Put $z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{\frac{4-3z}{z-2}}{z(z-1)}\,dz = -\int \frac{\frac{4-3z}{z-2}}{z\cancel{(z-1)}}\,dz + \int_C \frac{\frac{4-3z}{z-2}}{z-1}\,dz$$

$$= -2\pi i f(0) + 2\pi i f(1)$$

$$= -2\pi i\left(\frac{4}{-2}\right) + 2\pi i\left(\frac{1}{-1}\right) = 4\pi i - 2\pi i = 2\pi i$$

9) Evaluate $\int_C \frac{z\,dz}{(z-1)(z-2)^2}$ where C is $|z-2| = \frac{1}{2}$ using cauchy's integral formula.

Soln :

$|z-2| = |1-2| = |-1| = 1 > 1/2$

$\therefore z = 1$ lies outside C.

$|z-2| = |2-2| = 0 < 1/2$

$\therefore z = 2$ lies inside C.

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}\,dz$$

$$= 2\pi i\, f'(2)$$

$$= 2\pi i \left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \left[\frac{1-2}{1^2}\right] = 2\pi i(-1) = -2\pi i.$$

Taylor's Series :

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

This is known as Taylor's series of $f(z)$ at $z = a$.

Maclaurin's Series :

Put $a = 0$ in the Taylor series for $f(z)$ then

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

This series is called Maclaurin's series of $f(z)$.

Problems :

1) **Expand $f(z) = \sin z$ in a Taylor series about $z = 0$.**

Soln :

Function	At $z = 0$
$f(z) = \sin z$	$f(0) = 0$
$f'(z) = \cos z$	$f'(0) = 1$
$f''(z) = -\sin z$	$f''(0) = 0$
$f'''(z) = -\cos z$	$f'''(0) = -1$
$f^{IV}(z) = \sin z$	$f^{IV}(0) = 0$

$$= 0 + \frac{1 \cdot x}{1!} + \frac{0 \cdot x^2}{2!} + \frac{(-1)x^3}{3!} + 0 + \cdots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \cdots$$

2) Expand $\frac{x-1}{x+1}$ in Taylor series about the point $x = 1$.

Soln:

Function	At $x = 1$
$f(x) = \frac{x-1}{x+1}$	$f(1) = 0$
$f'(x) = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$	$f'(1) = 1/2$
$f''(x) = \frac{-4}{(x+1)^3}$	$f''(1) = -1/2$
$f'''(x) = \frac{12}{(x+1)^4}$	$f'''(1) = 3/4$

Taylor series about $x = 1$

$$f(x) = f(1) + (x-1)\frac{f'(1)}{1!} + \frac{(x-1)^2}{2!} f''(1) + \cdots$$

$$= 0 + \frac{(x-1)}{1!}\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{6}\left(\frac{3}{4}\right) + \cdots$$

$$= \frac{x-1}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} + \cdots$$

3) Expand $\log(1+x)$ as a Taylor's series about $x = 0$.

Soln:

Function	At $x = 0$.
$f(x) = \log(1+x)$	$f(0) = \log 1 = 0$
$f'(x) = \frac{1}{1+x}$	$f'(0) = 1$.

$$f'''(z) = \frac{2}{(1+z)^3} \qquad f'''(0) = 2$$

$$f^{IV}(z) = \frac{-6}{(1+z)^4} \qquad f^{IV}(0) = -6$$

Taylor's series about $z = 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$= 0 + \frac{1}{1!} z + \frac{(-1)}{2} z^2 + \frac{2}{6} z^3 + \frac{(-6)}{24} z^4 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Laurent's Series :

Let C_1, C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic on C_1 & C_2 and in the annular region R. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{1+n}}\, dz$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}}\, dz.$$

Problems :

1) Find the Laurent's series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ valid in the region

(i) $1 < |z| < 2$ (ii) $|z| > 2$ and $0 < |z-1| < 1$.

Soln :

Given, $f(z) = \frac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(ii) Given $|z| > 2 \Rightarrow 2 < |z|$

$$\Rightarrow 1 > \frac{2}{|z|} \Rightarrow \frac{2}{|z|} < 1$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z\left(1-1/z\right)} + \frac{1}{z\left(1-2/z\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z) = -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

(i) $1 < |z| < 2$

$|z| < 2 \quad , \quad 1 < |z|$

$\frac{|z|}{2} < 1 \quad , \quad \frac{1}{|z|} < 1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

(iii) Given $0 < |z-1| < 1$

Let $u = z-1$

$0 < |u| < 1 \Rightarrow |u| < 1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{u} + \frac{1}{u-1} = -\frac{1}{u} - \frac{1}{1-u} = -\frac{1}{u} - (1-u)^{-1}$$

$$= -\frac{1}{u} - \left[1+u+u^2+u^3+\cdots\right]$$

Put $u = z-1$

$$f(z) = -\frac{1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$$

Soln :

Given, $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$1 = A(z+3) + B(z+1)$.

Put $z=-3 \Rightarrow B = -1/2$

Put $z=-1 \Rightarrow A = 1/2$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}.$$

(i) $|z| > 3 \Rightarrow 1 > \dfrac{3}{|z|} \Rightarrow \dfrac{3}{|z|} < 1$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$- \frac{1}{2z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

(ii) $1 < |z| < 3$

$1 < |z| \qquad |z| < 3$

$\dfrac{1}{|z|} < 1 \qquad \dfrac{|z|}{3} < 1$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{6\left(\frac{z}{3}+1\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

Soln :

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3)+B(z+3)+C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3)+B(z+3)+C(z+2).$$

Put $z=-2 \Rightarrow B=3$

Put $z=-3 \Rightarrow C=-8$

Put $z=0 \Rightarrow -1 = 6A+3B+2C$

$\Rightarrow -1 = 6A+9-16$

$\Rightarrow A=1$

$$f(z) = 1+\frac{3}{z+2}-\frac{8}{z+3}.$$

(i) $2<|z|<3$

$|z|<3 \qquad 2<|z|$

$\frac{|z|}{3}<1 \qquad \frac{2}{|z|}<1$

$$f(z) = 1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\cdots\right]$$

(ii) $|z|>3 \Rightarrow 3<|z| \Rightarrow \frac{3}{|z|}<1$

$$f(z) = 1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

Soln:

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$= \frac{A(z-2)(z+1) + Bz(z+1) + Cz(z-2)}{z(z-2)(z+1)}.$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z=0 \Rightarrow A=1$

Put $z=2 \Rightarrow B=2$

Put $z=-1 \Rightarrow C=-3$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given, $1 < |z+1| < 3$.

put $z+1=u$

$1 < |u| < 3$

$1 < |u|$ $\quad\quad$ $|u| < 3$

$\frac{1}{|u|} < 1$ $\quad\quad$ $\frac{|u|}{3} < 1$.

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{u-1} + \frac{2}{u-1-2} - \frac{3}{u-1+1} = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= -\frac{3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{3\left(\frac{u}{3}-1\right)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^2+\left(\frac{1}{u}\right)^3+\cdots\right] - \frac{2}{3}\left[1+\left(\frac{u}{3}\right) + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3+\cdots\right]$$

$$= -\frac{3}{u} + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \left(\frac{1}{u}\right)^4 + \cdots - \frac{2}{3}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\left(\frac{u}{3}\right)^3+\cdots\right]$$

$$-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3 \cdots\right]$$

Singularities :

The point $z = a$ at which the function $f(z)$ is not analytic is called a singular point.

Ex:

Let $f(z) = \frac{1}{z-2}$

Here $z = 2$ is a singular point.

Types of singularities

i) Isolated singularities

The point $z = a$ is said to be isolated singularity if the neighbourhood of $z = a$ contains no other singularity.

Ex:

Let $f(z) = \frac{1}{z-3}$

The function is not analytic only at $z = 3$.

$z = 3$ is an isolated singularities.

ii) Removable singularity

A point $z = a$ is called a removable singularities of $f(z)$ if $\lim_{z \to a} f(z) = a$, finite

Ex:

$f(z) = \frac{\tan z}{z}$

Here $z = 0$ is a singular point

$\lim_{z \to 0} \frac{\tan z}{z} = \frac{\tan 0}{0} = \frac{0}{0}$ (indefinite finite)

Applying L-Hospital rule

$\lim_{z \to 0} \frac{\tan z}{z} = \lim_{z \to 0} \frac{\sec^2 z}{1} = \sec^2 0 = \frac{1}{\cos^2 0} = 1.$

$\lim_{z \to 0} \frac{\tan z}{z} = 1$ (finite value)

$$\lim_{z \to a} (z-a)^n f(z) \neq 0$$

If $n=1$ it is called a simple pole.

If $n=2$, it is called a double pole.

Ex:

$$f(z) = \frac{1}{(z-1)(z+2)^2}$$

The singularities are $1, -2$

Take $z=1$

$$\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{1}{(z-1)(z+2)^2} = \frac{1}{9} \neq 0$$

$z=1$ is a simple pole.

Take $z=-2$

$$\lim_{z \to -2} (z+2) f(z) = \lim_{z \to -2} (z+2)^2 \frac{1}{(z-1)(z+2)^2}$$

$$= \frac{-1}{3} \neq 0$$

$\therefore z=-2$ is a pole of order 2 (double pole)

Essential Singularity

A singular point $z=a$ is said to be an essential singular point of $f(z)$ if the Laurent series of $f(z)$ about $z=a$ possess of infinite number of terms in the principal part.

Ex:

$$f(z) = e^{\frac{1}{z-1}}$$

Here $z=1$ is a singular point

At $z=1$, $f(z) = e^{1/0} = e^{\infty}$ (which is not defined)

Also $z=1$ is not a pole (or) removable singularity

$\therefore z=1$ is an essential singularity

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

The co-efficient of b of $\frac{1}{z-a}$ is called Residue of $f(z)$ at $z = a$.

i) If $z = a$ is a simple pole then

$$\underset{z=a}{\text{Res}}\, f(z) = \lim_{z \to a} (z-a) f(z).$$

ii) If $z = a$ is a pole of order n then

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

Problems under Residues:

1) Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Soln:

Here $z = 2$ is a simple pole.

$$\underset{z \to a}{\text{Res}}\, f(z) = \lim_{z \to a} (z-a) f(z)$$

$$\underset{z \to 2}{\text{Res}}\, f(z) = \lim_{z \to 2} (z-2) \frac{4}{z^3(z-2)} = \frac{4}{2^3} = \frac{1}{2}$$

2) Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Soln:

$$f(z) = \frac{e^{2z}}{(z+1)^2}$$

$z = -1$ is a pole of order 2.

$$\underset{z \to a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

$$\underset{z \to -1}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz} \left\{ (z+1)^2 \frac{e^{2z}}{(z+1)^2} \right\}$$

$$= \lim_{z \to -1} 2e^{2z} = 2e^{-2}$$

Soln :

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

$z = 1$ is a pole of order 2.

$z = -2$ is a pole of order 1.

$$\underset{z \to a}{\text{Res}}\, f(z) = \underset{z \to a}{\text{lt}}\, (z-a) f(z).$$

$$\underset{z \to -2}{\text{Res}}\, f(z) = \underset{z \to -2}{\text{lt}}\, (z+2) \frac{z^2}{(z-1)^2(z+2)} = \frac{4}{9}$$

$$\underset{z \to a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \underset{z \to a}{\text{lt}} \frac{d^n}{dz^{n-1}} (z-a)^n f(z)$$

$$\underset{z \to 1}{\text{Res}}\, f(z) = \frac{1}{1!} \underset{z \to 1}{\text{lt}} \frac{d}{dz} (z-1)^2 \frac{z^2}{(z-1)^2(z+2)}$$

$$= \underset{z \to 1}{\text{lt}} \frac{d}{dz} \frac{z^2}{z+2}$$

$$= \underset{z \to 1}{\text{lt}} \frac{(z+2)\,2z - z^2(1)}{(z+2)^2} = \frac{3(2)-1}{3^2} = \frac{5}{9}$$

Residue using Laurent series :

$\left[\text{Res } f(z)\right]_{z=a}$ = co-efficient of $\frac{1}{z-a}$ in the Laurent series of $f(z)$ about $z = a$.

Problems :

1) Obtain the Laurent expansion of the function $\frac{e^z}{(z-1)^2}$.

Soln :

$$f(z) = \frac{e^z}{(z-1)^2}$$

$z = 1$ is a pole of order 2.

Put $z - 1 = u$.

$z = u + 1$

$$f(z) = \frac{e^{u+1}}{u^2} = \frac{e^u \cdot e}{u^2}$$

$$= \frac{e}{u^2}\left[1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right]$$

$$= \frac{0}{(z-1)^2} + \frac{0}{1!(z-1)} + \frac{0}{2!} + \frac{0(z-1)}{3!} + \dots$$

This is the Laurent series expansion of $f(z)$ about $z=1$.

$[\text{Res } f(z)]_{z=1}$ = co-eff. of $\frac{1}{z-1} = \frac{0}{1!} = 0$.

2) Find the residues of $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansion.

Soln:

$$f(z) = \frac{z^2}{(z-1)(z+2)^2}$$

$z=1$ and $z=-2$ are isolated singularities of $f(z)$.

$$f(z) = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}$$

$$z^2 = A(z+2)^2 + B(z-1)(z+2) + C(z-1).$$

Put $z=1 \Rightarrow A = 1/9$

Put $z=-2 \Rightarrow C = -4/3$

Put $z=0 \Rightarrow B = 8/9$

$$f(z) = \frac{1}{9(z-1)} + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2} \longrightarrow (1)$$

Case (i):

To find: $[\text{Res } f(z)]_{z=1}$

Put $z-1=u \Rightarrow z = u+1$

$$(1) \Rightarrow f(z) = \frac{1}{9u} + \frac{8}{9(u+1+2)} - \frac{4}{3(u+1+2)^2}$$

$$= \frac{1}{9u} + \frac{8}{9(u+3)} - \frac{4}{3(u+3)^2}$$

$$= \frac{1}{9u} + \frac{8}{9(3)\left(1+\frac{u}{3}\right)} - \frac{4}{3\cdot 9\left(1+\frac{u}{3}\right)^2}$$

$$= \frac{1}{9u} + \frac{8}{27}\left(1+\frac{u}{3}\right)^{-1} - \frac{4}{27}\left(1+\frac{u}{3}\right)^{-2}$$

$$= \frac{1}{9u} + \frac{8}{27}\left[1 - \frac{u}{3} + \left(\frac{u}{3}\right)^2 - \left(\frac{u}{3}\right)^3 + \dots\right]$$

$$- \frac{4}{27}\left[1 - 2\left(\frac{u}{3}\right) + 3\left(\frac{u}{3}\right)^2 - 4\left(\frac{u}{3}\right)^3 + \dots\right]$$

$$-\frac{4}{27}+\frac{8}{27}\left(\frac{u}{3}\right)-\frac{12}{27}\left(\frac{u}{3}\right)^2+\frac{16}{27}\left(\frac{u}{3}\right)^3+\cdots$$

$$=\frac{1}{9u}+\frac{4}{27}-\frac{4}{27}\left(\frac{u}{3}\right)^2+\frac{8}{27}\left(\frac{u}{3}\right)^3+\cdots$$

$$=\frac{1}{9(z-1)}+\frac{4}{27}-\frac{4}{27}\left(\frac{z-1}{3}\right)^2+\frac{8}{27}\left(\frac{z-1}{3}\right)^3+\cdots \rightarrow (2)$$

$[\text{Res } f(z)]_{z=1}$ = co-eff. of $\frac{1}{z-1}=\frac{1}{9}$

case (ii) : To find $[\text{Res } f(z)]_{z=-2}$

Put $z+2=u \Rightarrow z=u-2$

$$(2)\Rightarrow f(z)=\frac{1}{9(u-2-1)}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$=\frac{1}{9(u-3)}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$=\frac{1}{-27\left(1-\frac{u}{3}\right)}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$=-\frac{1}{27}\left(1-\frac{u}{3}\right)^{-1}+\frac{8}{9u}-\frac{4}{3u^2}$$

$$=-\frac{1}{27}\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\left(\frac{u}{3}\right)^3+\cdots\right]+\frac{8}{9(z+2)}-\frac{4}{3(z+2)^2}$$

$$=-\frac{1}{27}\left[1+\left(\frac{z+2}{3}\right)+\left(\frac{z+2}{3}\right)^2+\left(\frac{z+2}{3}\right)^3+\cdots\right]+\frac{8}{9(z+2)}-\frac{4}{3(z+2)^2}$$

$[\text{Res } f(z)]_{z=-2}$ = co-eff. of $\frac{1}{z+2}=\frac{8}{9}$

Contour Integration :

Type - I :

$$\int_0^{2\pi} f(\theta)\,d\theta \;, \quad |z|=1$$

Take $z=e^{i\theta} \Rightarrow d\theta=\frac{dz}{iz}$

Replace $\cos\theta$ by $\frac{z^2+1}{2z}$

$\sin\theta$ by $\frac{z^2-1}{2iz}$

Soln:

Take $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$\cos\theta$ by $\frac{z^2+1}{2z}$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{dz}{iz\left[a+b\left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{bz^2+2az+b}{2z}\right]}$$

$$= \frac{2}{bi}\int_C \frac{dz}{z^2+\frac{2a}{b}z+1}$$

Let $f(z) = \frac{1}{z^2+\frac{2a}{b}z+1} = \frac{1}{(z-\alpha)(z-\beta)}$

$z^2+\frac{2a}{b}z+1 = 0$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-\frac{4b^2}{b^2}}}{2} = \frac{-\frac{2a}{b} \pm \frac{2\sqrt{a^2-b^2}}{b}}{2}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b} = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad \frac{-a-\sqrt{a^2-b^2}}{b}$$

$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

Given $a > b > 0$

$a = 2$, $b = 1$

$\alpha = -2+\sqrt{4-1} = -2+1.732 = -0.268 < 1$.

$|z| = |-0.268| = 0.268 < 1$.

$\therefore z = \alpha$ lies inside C.

$\beta = \frac{-a-\sqrt{a^2-b^2}}{b} = -2-\sqrt{4-1} = -2-1.732 = -3.732 \nless 1$

$|z| = |-3.732| = 3.732 > 1$.

$\therefore z = \beta$ lies outside C.

$$= \frac{1}{\alpha - \beta}$$

$$= \frac{1}{\left[\frac{-a+\sqrt{a^2-b^2}+a+\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{b}{2\sqrt{a^2-b^2}}$$

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues) .}$$

$$= 2\pi i \cdot \frac{b}{2\sqrt{a^2-b^2}} = \frac{\pi i b}{\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{ib} \cdot \frac{\pi i b}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Note :

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

i) $$\int_0^{2\pi} \frac{d\theta}{13+5\cos\theta} = \frac{2\pi}{\sqrt{13^2-5^2}} = \frac{2\pi}{\sqrt{144}} = \frac{2\pi}{12} = \frac{\pi}{6}.$$

ii) $$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{4-1}} = \frac{2\pi}{\sqrt{3}}$$

iii) $$\int_0^{2\pi} \frac{d\theta}{5+\cos\theta} = \frac{2\pi}{\sqrt{5^2-1^2}} = \frac{2\pi}{\sqrt{24}}$$

2) Using contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, $a>b>0$.

Soln :

Take $z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$$\sin\theta = \frac{z^2-1}{2iz}$$

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_C^{2\pi} \frac{dz}{iz\left[a+b\left(\frac{z^2-1}{2iz}\right)\right]}$$

$$= \int_C \frac{dz}{iz\left[\frac{2iaz+b(z^2-1)}{2iz}\right]}$$

$$= \frac{2}{b} \int_c \frac{dz}{z^2+\left(\frac{2ia}{b}\right)z-1}$$

$I = \frac{2}{b}\, 2\pi i$ (Sum of the residues)

$$f(z) = \frac{1}{z^2+\left(\frac{2ia}{b}\right)z-1}$$

$$z^2+\left(\frac{2ia}{b}\right)z-1=0$$

$$z = \frac{-\frac{2ia}{b} \pm \sqrt{\frac{-4a^2}{b^2}+4}}{2} = \frac{-\frac{2ia}{b} \pm \frac{i2\sqrt{a^2-b^2}}{b}}{2}$$

$$= \frac{-ia \pm i\sqrt{a^2-b^2}}{b} = i\left[\frac{-a \pm \sqrt{a^2-b^2}}{b}\right]$$

$$z = i\left[\frac{-a+\sqrt{a^2-b^2}}{b}\right],\ i\left[\frac{-a-\sqrt{a^2-b^2}}{b}\right]$$

$= \alpha, \beta$.

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

Given, $a > b > 0$

$a = 2,\ b = 1$

$$z = \alpha = i\left[\frac{-2 \pm \sqrt{4-1}}{1}\right] = i(-2+\sqrt{3}) = i(0.268)$$

$|z| = |i(0.268)| = 0.268 < 1$.

$\therefore z = \alpha$ lies inside C.

$$z = \beta = i\left[\frac{-2-\sqrt{4-1}}{1}\right] = i(-2-1.732) = i(-3.732)$$

$|z| = |i(-3.732)| = 3.732 > 1$.

$\therefore z = \beta$ lies outside C.

$$\operatorname*{Res}_{z\to\alpha} f(z) = \lim_{z\to\alpha} (z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}$$

$$= \frac{1}{\alpha-\beta}$$

$$\lim_{z \to \alpha} f(z) = \frac{\quad}{2i\sqrt{a^2-b^2}}$$

$$I = \frac{2}{b} \times 2\pi i \times \frac{b}{2i\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Note :

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{\sqrt{25-16}} = \frac{2\pi}{\sqrt{9}} = \frac{2\pi}{3}.$$

$$\int_0^{2\pi} \frac{d\theta}{5+\sin\theta} = \frac{2\pi}{\sqrt{25-1}} = \frac{2\pi}{\sqrt{24}} = \frac{2\pi}{2\sqrt{6}} = \frac{\pi}{\sqrt{6}}$$

3) Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$.

Soln :

Let $z = e^{i\theta}$ $\quad \frac{dz}{iz} = d\theta$.

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\cos 2\theta = R\cdot P \text{ of } e^{2i\theta}$$
$$= R\cdot P \text{ of } z^2$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = R\cdot P \text{ of } \int_C \frac{z^2}{5+4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$= R\cdot P \text{ of } \int_C \frac{z^2}{2z^2+5z+2} \frac{dz}{i}$$

$$= R\cdot P \text{ of } \frac{1}{2i} \int_C \frac{z^2}{z^2+\frac{5}{2}z+1}\, dz .$$

$$= R\cdot P \text{ of } \frac{1}{2i} \times 2\pi i \times \text{sum of residues}$$

$$f(z) = \frac{z^2}{z^2+\frac{5}{2}z+1}$$

$$z = \frac{\ldots 2 - \sqrt{\ldots 4}}{2} = \frac{\ldots}{2}$$

$$= \frac{-\frac{5}{2} + \frac{3}{2}}{2}, \quad \frac{-\frac{5}{2} - \frac{3}{2}}{2} = -\frac{1}{2}, -2$$

$z = -\frac{1}{2}$

$|z| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

$\therefore z = -\frac{1}{2}$ lies inside C.

$z = -2$

$|z| = |-2| = 2 > 1$

$\therefore z = -2$ lies outside C.

$$\operatorname*{Res}_{z \to -\frac{1}{2}} f(z) = \operatorname*{lt}_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}$$

$$= \frac{\left(-\frac{1}{2}\right)^2}{-\frac{1}{2}+2} = \frac{(1/4)}{(3/2)} = \frac{1}{6}$$

$$I = R \cdot P \text{ of } \frac{1}{2i} \times 2\pi i \times \frac{1}{6} = R \cdot P \text{ of } \frac{\pi}{6} = \frac{\pi}{6}.$$

4) Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

<u>Soln</u>:

Put $z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z^2+1}{2z}$

$\cos 3\theta = R \cdot P$ of $e^{i3\theta}$

$= R \cdot P$ of $(e^{i\theta})^3 = R \cdot P$ of z^3.

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = R \cdot P \text{ of } \int_0^{2\pi} \frac{z^3}{5-4\left[\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz}$$

$$= R \cdot P \text{ of } \int_C \frac{z^3}{\left(\frac{10z - 4z^2 - 4}{2z}\right)} \frac{dz}{iz}$$

$$= R.P \text{ of } \frac{2}{i} \int_c \frac{z^3}{-4\left[z^2 - \frac{5}{2}z + 1\right]} dz$$

$$= - R.P \text{ of } \frac{1}{2i} \int_c \frac{z^3}{z^2 - \frac{5}{2}z + 1} dz .$$

$$= - R.P \text{ of } \frac{1}{2i} \times 2\pi i \times \text{ sum of residues}$$

$$f(z) = \frac{z^3}{z^2 - \frac{5}{2}z + 1}$$

$$z^2 - \frac{5}{2}z + 1 = 0 .$$

$$z = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{\frac{5}{2} + \frac{3}{2}}{2}, \frac{\frac{5}{2} - \frac{3}{2}}{2}$$

$$= 2, 1/2$$

$z = 2$

$|z| = |2| = 2 > 1 .$

$\therefore z = 2$ lies outside c.

$z = \frac{1}{2}$

$|z| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$

$\therefore z = \frac{1}{2}$ lies inside c.

$$\underset{z \to \frac{1}{2}}{\text{Res}} f(z) = \left(z - \frac{1}{2}\right) \frac{z^3}{\left(z - \frac{1}{2}\right)(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{\left(-3/2\right)} = \frac{1}{8} \times \frac{2}{-3} = -\frac{1}{12}$$

$$I = - R.P \text{ of } \frac{1}{2i} \times 2\pi i \times -\frac{1}{12}$$

$$= - R.P \text{ of } \left(\frac{-\pi}{12}\right)$$

$$= R.P \text{ of } \frac{\pi}{12}$$

$$I = \pi/12 .$$

Take $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= R\cdot P \text{ of } \left[\frac{1-e^{i2\theta}}{2}\right] = R\cdot P \text{ of } \left[\frac{1-z^2}{2}\right]$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = R\cdot P \text{ of } \int_c \frac{\left(\frac{1-z^2}{2}\right)}{a+b\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= R\cdot P \text{ of } \frac{1}{2i} \int_c \frac{2(1-z^2)}{2az+bz^2+b}\, dz$$

$$= R\cdot P \text{ of } \frac{1}{i} \int_c \frac{1-z^2}{bz^2+2az+b}\, dz$$

$$= R\cdot P \text{ of } \frac{1}{i} \int_c \frac{(1-z^2)}{b\left[z^2+\frac{2a}{b}z+1\right]}\, dz$$

$$= R\cdot P \text{ of } \frac{1}{bi} \int_c \frac{1-z^2}{z^2+\frac{2a}{b}z+1}\, dz$$

$$= R\cdot P \text{ of } \frac{1}{bi} \times 2\pi i \times \text{sum of residues}$$

$$f(z) = \frac{1-z^2}{z^2+\frac{2a}{b}z+1}$$

$$z^2 + \frac{2a}{b}z + 1 = 0.$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$= \frac{-\frac{2a}{b} \pm \frac{2\sqrt{a^2-b^2}}{b}}{2} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$= \frac{-a+\sqrt{a^2-b^2}}{b},\ \frac{-a-\sqrt{a^2-b^2}}{b}$$

~~z~~ Given, $a > b$.

$|z| = |-0.268| = 0.268 < 1$.

$\therefore z = \alpha$ lies inside C.

$z = \beta = -2 - \sqrt{3} = -3.732$

$|z| = |-3.732| = 3.732 > 1$.

$\therefore z = \beta$ lies outside C.

$$\underset{z \to \alpha}{\text{Res}}\, f(z) = \lim_{z \to \alpha} (z-\alpha) \frac{1 - z^2}{(z-\alpha)(z-\beta)}$$

$$= \frac{1-\alpha^2}{\alpha - \beta} = \frac{1 - \left(\frac{-a + \sqrt{a^2-b^2}}{b}\right)^2}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1 - \left[\frac{a^2 + a^2 - b^2 - 2a\sqrt{a^2-b^2}}{b^2}\right]}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{2b^2 - 2a^2 + 2a\sqrt{a^2-b^2}}{2b\sqrt{a^2-b^2}}$$

$$= \frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}}$$

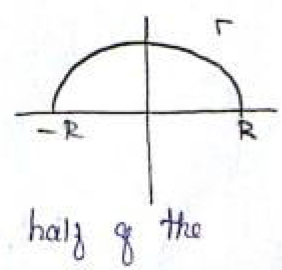
$I =$ R.P of $\frac{1}{bi} \times 2\pi i \times$ sum of residues

$= $ R.P of $\frac{1}{bi} \times 2\pi i \times \frac{b^2 - a^2 + a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}}$

$$= \frac{2\pi \left[b^2 - a^2 + a\sqrt{a^2-b^2}\right]}{b^2\sqrt{a^2-b^2}}$$

Type - 2 :

$$\int_{-\infty}^{\infty} f(x)\, dx$$



consider c which consists of the upper half of the semi circle.

As $R \to \infty$, $\int_\Gamma f(z)\,dz = 0$.

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz.$$

Problems :

1) P.T $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$, $a>0$, $b>0$.

Soln :

Consider C which consist of the upper half of the semi circle.

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

As $R \to \infty$

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz$$

$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$(z^2+a^2)(z^2+b^2) = 0$$

$$z^2 = -a^2 \qquad z^2 = -b^2$$

$$z = \pm ai \qquad z = \pm bi$$

$z = -ai, -bi$ lies outside C.

$z = ai, bi$ lies inside C.

$$\operatorname*{Res}_{z\to ai} f(z) = \lim_{z\to ai} (z-ai)\frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{(ai)^2}{(ai+ai)[(ai)^2+b^2]} = \frac{-a^2}{2ai\,(b^2-a^2)}$$

$$= \frac{a}{2i\,(a^2-b^2)}$$

$$\operatorname*{Res}_{z\to bi} f(z) = \lim_{z\to bi} (z-bi)\frac{z^2}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

$\int_C f(z)\,dz = 2\pi i \times$ sum of the residue

$$= 2\pi i\left[\frac{a-b}{2i(a^2-b^2)}\right] = \pi\left[\frac{a-b}{(a+b)(a-b)}\right]$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}.$$

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$$

Note :

i) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)} = \frac{\pi}{2(3+2)} = \frac{\pi}{10}$

ii) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)}$

iii) $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$

iv) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{(1+2)} = \frac{\pi}{3}.$

2) Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

Soln :

$$f(x) = \frac{1}{(x^2+a^2)^2}$$

$$f(z) = \frac{1}{(z^2+a^2)^2}$$

$(z^2+a^2)^2 = 0 \Rightarrow z^2+a^2 = 0 \Rightarrow z^2 = -a^2 \Rightarrow z = \pm ia$

$$f(z) = \frac{1}{(z+ia)^2(z-ia)^2}$$

$z = -ia, ia$ is a pole of order 2.

$z = ia$ lies in the upper half of the plane & lies inside

$$= \lim_{z \to ia} \frac{d}{dz} \frac{1}{(z+ia)^2} = \lim_{z \to ia} \frac{d}{dz} (z+ia)^{-2}$$

$$= \lim_{z \to ia} (-2)(z+ia)^{-3} = (-2)(2ia)^{-3}.$$

$$= \frac{-2}{(2ia)^3} = \frac{-2}{-8ia^3} = \frac{1}{4ia^3}$$

By cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of residues)}$$

$$= 2\pi i \frac{1}{4ia^3} = \frac{\pi}{2a^3}$$

$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx \longrightarrow (1)$$

By cauchy's lemma, $R \to \infty$ $\int_\Gamma f(z)\,dz \longrightarrow 0$

$$\frac{\pi}{2a^3} = 0 + \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

$$2\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.$$

3) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{12}$

Soln:

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

As $R \to \infty$

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx.$$

$$f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$$

$$z^4+10z^2+9 = 0 \Rightarrow (z^2+1)(z^2+9) = 0.$$

$z = \pm i \qquad z = \pm 3i$

$z = i, 3i$ lies inside C.

$z = -i, -3i$ lies outside C.

$$\underset{z \to i}{\text{Res}}\, f(z) = \underset{z \to i}{\text{lt}} (z-i) \frac{z^2 - z + 2}{(z+i)(z-i)(z^2+9)}$$

$$= \frac{-1-i+2}{2i(-1+9)} = \frac{1-i}{16i}$$

$$\underset{z \to 3i}{\text{Res}}\, f(z) = \underset{z \to 3i}{\text{lt}} (z-3i) \frac{z^2 - z + 2}{(z^2+1)(z+3i)(z-3i)}$$

$$= \frac{-9-3i+2}{(-9+1)\,6i} = \frac{-3i-7}{-8(6i)} = \frac{3i+7}{48i}$$

$$\int_C f(z)\,dz = 2\pi i \left[\frac{1-i}{16i} + \frac{3i+7}{48i}\right]$$

$$= 2\pi i \left[\frac{3-3i+3i+7}{48i}\right] = 2\pi\left(\frac{10}{48}\right) = \frac{5\pi}{12}$$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \frac{5\pi}{12}.$$

Type - 3 :

$$\int_{-\infty}^{\infty} f(x) \cos mx\, dx \quad (\text{or}) \quad \int_{-\infty}^{\infty} f(x) \sin mx\, dx.$$

Problems :

1) Solve $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\,dx = \frac{\pi}{2a} e^{-ma}, \; m > 0.$

Soln :

$$\int_0^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx.$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2}\,dx = \text{R.P of} \int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx.$$

Consider C is the upper half of the semi-circle.

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{-\infty} f(x)\,dx$$

Let $f(z) = \dfrac{e^{imz}}{z^2+a^2}$

$z^2+a^2=0 \Rightarrow z^2=-a^2 \Rightarrow z=\pm ai$

$z=ai$ lies inside C

$z=-ai$ lies outside C.

$$\operatorname*{Res}_{z\to ai} f(z) = \lim_{z\to ai}(z-ai)\frac{e^{imz}}{(z+ai)(z-ai)}$$

$$= \frac{e^{-am}}{2ai}$$

$$\int_C f(z)\,dz = 2\pi i\cdot\frac{e^{-am}}{2ai} = \frac{\pi e^{-am}}{a}$$

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi e^{-am}}{a}$$

$$\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2}\,dx = \frac{1}{2}\,\frac{\pi e^{-am}}{a} = \frac{\pi e^{-am}}{2a}$$

2) Evaluate $\displaystyle\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx$, $m>0$, $a>0$ by contour integration.

Soln:

$$\int_0^\infty \frac{x\sin mx}{x^2+a^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin mx}{x^2+a^2}\,dx$$

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^2+a^2}\,dx = \text{I.P of } \int_{-\infty}^{\infty}\frac{x\,e^{imx}}{x^2+a^2}\,dx$$

Consider C is the upper half of the semi-circle

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

As $R\to\infty$, $\int_\Gamma f(z)\,dz = 0$.

$$\int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx.$$

$$f(z) = \frac{z\,e^{imz}}{z^2+a^2}$$

$z^2+a^2=0 \Rightarrow z^2=-a^2 \Rightarrow z=\pm ai$

$$\underset{z \to ai}{\text{Res}} f(z) = \lim_{z \to ai} (z-ai) \frac{z e^{imz}}{(z+ai)(z-ai)}$$

$$= \frac{ai \cdot e^{im(ai)}}{2ai} = \frac{e^{-am}}{2}$$

$$\int_c f(z)\,dz = 2\pi i \times \frac{e^{-am}}{2} = \pi i e^{-am}$$

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \text{I.P of } \left[\pi i e^{-am}\right]$$

$$= \pi e^{-am}$$

$$\int_{0}^{\infty} \frac{x \cdot \sin mx}{x^2+a^2}\,dx = \frac{\pi e^{-am}}{2}$$