The vector differential operator is $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

1) Define gradient of Scalar function

$$\text{grad } \varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

grad φ $\rightarrow$ Normal vector to the Surface φ

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(1) Find grad φ at $(2,0,2)$ if $\varphi = x^2+y^2+z^2-8$

Sol

Given $\varphi = x^2+y^2+z^2-8$

$$\frac{\partial\varphi}{\partial x} = 2x \quad \bigg| \quad \frac{\partial\varphi}{\partial y} = 2y \quad \bigg| \quad \frac{\partial\varphi}{\partial z} = 2z$$

$$\text{grad } \varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

At $(2,0,2)$

$$\text{grad } \varphi = 4\vec{i} + 0\vec{j} + 4\vec{k}$$

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(2) Find grad f at $(1,0,-1)$ if $f = xyz$

Sol

Given $f = xyz$

$$\frac{\partial f}{\partial x} = yz \quad \bigg| \quad \frac{\partial f}{\partial y} = xz \quad \bigg| \quad \frac{\partial f}{\partial z} = xy$$

$$\text{grad } f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2$.

$2r \frac{\partial r}{\partial x} = 2x$ | $2r \frac{\partial r}{\partial y} = 2y$ | $2r \frac{\partial r}{\partial z} = 2z$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ | $\frac{\partial r}{\partial y} = \frac{y}{r}$ | $\frac{\partial r}{\partial z} = \frac{z}{r}$.

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(3) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, Prove that $\nabla r^n = n.r^{n-2}\vec{r}$

Sol

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = x^2+y^2+z^2$; $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\nabla r^n = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, n.r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n.r^{n-1}\frac{\partial r}{\partial y} + \vec{k}.n.r^{n-1}\frac{\partial r}{\partial z}$$

$$= \vec{i}\, n r^{n-1}\cdot\frac{x}{r} + \vec{j}\, n r^{n-1}\frac{y}{r} + \vec{k}\, n r^{n-1}\frac{z}{r}$$

$$= \vec{i}\, n r^{n-2} x + \vec{j}\, n r^{n-2} y + \vec{k}\, n r^{n-2} z$$

$$= n r^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$\nabla r^n = n r^{n-2}\vec{r}$$

Note

Surface φ ... $\frac{}{|\nabla\varphi|}$

(ii) Normal derivative $= |\nabla\varphi|$

Find the unit vector normal to the surface $x^2y + 2xz^2 = 8$ at $(1,0,2)$.

Sol

Let $\varphi = x^2y + 2xz^2 - 8$

$\frac{\partial\varphi}{\partial x} = 2xy + 2z^2$ | $\frac{\partial\varphi}{\partial y} = x^2$ | $\frac{\partial\varphi}{\partial z} = 4xz$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = (2xy + 2z^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

At $(1,0,2)$

$$\nabla\varphi = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\varphi| = \sqrt{64+1+64} = \sqrt{129}.$$

Unit Normal vector $= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}$.

Find the unit normal vector to the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Sol

Let ...

(3) Find the normal derivative of $\varphi = xyz^2$ at (1,0

Sol

$$\varphi = xyz^2$$

$$\frac{\partial\varphi}{\partial x} = yz^2 \quad \Big| \quad \frac{\partial\varphi}{\partial y} = xz^2 \quad \Big| \quad \frac{\partial\varphi}{\partial z} = 2xyz$$

$$\text{grad } \varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

At (1,0,3)

$$\nabla\varphi = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

Normal derivative $= |\nabla\varphi| = \sqrt{0+81+0} = \sqrt{81} = 9.$

Note

Direction derivative $= \dfrac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$

Maximum derivative $= |\nabla\varphi|$

(7) Find the directional derivative of $F = xy$

at (1,1,1) in the direction of $\vec{i}+\vec{j}+\vec{k}$

Sol

Given $F = xyz$

$$\frac{\partial F}{\partial x} = yz \quad \Big| \quad \frac{\partial F}{\partial y} = xz \quad \Big| \quad \frac{\partial F}{\partial z} = xy$$

Directional derivative $= \dfrac{\nabla F \cdot \vec{a}}{|\vec{a}|} = \dfrac{3}{\sqrt{3}} = \sqrt{3}$.

Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i}+3\vec{j}+4\vec{k}$.

Sol

$$\varphi = 4xz^2 + x^2yz$$

$$\frac{\partial\varphi}{\partial x} = 4z^2 + 2xyz \quad \Big| \quad \frac{\partial\varphi}{\partial y} = x^2z \quad \Big| \quad \frac{\partial\varphi}{\partial z} = 8xz + x^2y$$

$$\nabla\varphi = \vec{i}(4z^2+2xyz) + \vec{j}(x^2z) + \vec{k}(8xz + x^2y)$$

At $(1,-2,1)$

$$\nabla\varphi = 0\vec{i} + \vec{j} + 6\vec{k}$$

$$\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k} \qquad |\vec{a}| = \sqrt{4+9+16} = \sqrt{29}.$$

$$D.D. = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|} = \frac{0+3+24}{\sqrt{29}} = \frac{27}{\sqrt{29}}$$

Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at $P(1,-2,-1)$ in the direction of PQ where $Q(3,-3,-2)$.

Sol

$$\nabla\varphi = \vec{i}(4z^2+2xyz) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

Max. $\quad D.D. = |\nabla\phi| = \sqrt{16+1+100} = \sqrt{117}$

Note

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

If $\vec{r}$ is the position vector of (x, y, z), P.T.

$$\nabla^2 r^n = n(n+1) r^{n-2}.$$

Sol

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$r^2 = x^2 + y^2 + z^2$; $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2 r^n = \frac{\partial^2(r^n)}{\partial x^2} + \frac{\partial^2(r^n)}{\partial y^2} + \frac{\partial^2(r^n)}{\partial z^2}.$$

$$\frac{\partial(r^n)}{\partial x} = n r^{n-1}\frac{\partial r}{\partial x} = n.r^{n-1}.\frac{x}{r} = n.r^{n-2}x$$

$$\frac{\partial^2(r^2)}{\partial x^2} = n\left[r^{n-2} + x(n-2)r^{n-3}.\frac{\partial r}{\partial x}\right]$$

$$= n.\left[r^{n-2} + x(n-2)r^{n-4}.x\right]$$

$$\boxed{\frac{\partial^2(r^2)}{\partial x^2} = n r^{n-2} + n(n-2) r^{n-4} x^2}$$

III^{ly}

$$\frac{\partial^2(r^2)}{\partial y^2} = n r^{n-2} + n(n-2) r^{n-4} y^2$$

$\partial^2(r^2)$ ... $n-4$ ...

$$\text{div}\ \vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

1) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, find div $\vec{r}$.

Sol

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{div}\ \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial (y)}{\partial y} + \frac{\partial (z)}{\partial z} = 1+1+1 = 3.$$

If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$ find grad (div $\vec{F}$).

Sol

$$\text{div}\ \vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$$

$$= 2x + 2z - x + z + 2z$$

$$= \text{div}\ \vec{F} = x + 5z$$

$$\text{grad}\ (\text{div}\ \vec{F}) = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

$$= \vec{i} + 0\vec{j} + 5\vec{k}.$$

Note

div $\vec{F} = 0 \Rightarrow \vec{F}$ is solenoidal.

Define Curl of a vector point function

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[0-0] = \vec{0}$$

If $\vec{F} = 3\vec{i} + x\vec{j} + y\vec{k}$, S.T. $\text{curl}(\text{curl } \vec{F}) = \vec{0}$.

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & x & y \end{vmatrix}$$

$$= \vec{i}[1-0] - \vec{j}[0-0] + \vec{k}[1-0]$$

$$= \vec{i} - 0\vec{j} + \vec{k}$$

$$\text{Curl}(\text{curl } \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 0 & 1 \end{vmatrix} = \vec{0}$$

Prove that $\text{curl}(\text{grad } \varphi) = \vec{0}$.

proof

$$\text{grad } \varphi = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

$$\vec{k}\left[\frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial^2\varphi}{\partial y\partial x}\right]$$

$$= 0\vec{i}-0\vec{j}+0\vec{k}=\vec{0}$$

Note

(i) Curl $\vec{F}=\vec{0} \Rightarrow \vec{F}$ is irrotational

(ii) $\vec{F}$ is irrotational $\Rightarrow \vec{F}=\nabla\varphi$

φ is called scalar potential

Is the position vector $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ irrotational?

Sol

Curl $\vec{r}=\vec{0}$

S.T. $\vec{F}=(y^2-z^2+3yz-2x)\vec{i}+(3xz+2xy)\vec{j}+(3xy-2xz+2z)\vec{k}$ is both solenoidal & irrotational.

Sol

$$\text{div }\vec{F}=\frac{\partial}{\partial x}(y^2-z^2+3yz-2x)+\frac{\partial}{\partial y}(3xz+2xy)+\frac{\partial}{\partial z}(3xy-2xz+2z)$$

$$=-2+2x-2x+2=0$$

$\Rightarrow \vec{F}$ is solenoidal

$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$

$\vec{F}$ is irrotational.

Find λ if $(2x+y)\vec{i} + (z-\lambda y)\vec{j} + (2\lambda z - x)\vec{k}$ is solenoidal.

Sol

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(2x+y) + \frac{\partial}{\partial y}(z-\lambda y) + \frac{\partial}{\partial z}(2\lambda z - x)$$

$$= 2 - \lambda + 2\lambda = 2 + \lambda$$

$\vec{F}$ is solenoidal $\Rightarrow \text{div } \vec{F} = 0 \Rightarrow 2+\lambda = 0 \Rightarrow \lambda$

Prove that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)$ irrotational & find the scalar potential φ s that $\vec{F} = \nabla\varphi$.

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}\left[-1+1\right] - \vec{j}\left[3z^2 - 3z^2\right] + \vec{k}\left[6x -\right.$$

$\text{Curl } \vec{F} = \vec{0} \Rightarrow \vec{F}$ is irrotational.

Let $\vec{F} = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$\frac{\partial\varphi}{\partial x} = 6xy + z^3$ | $\frac{\partial\varphi}{\partial y} = 3x^2 - z^2$ | $\frac{\partial\varphi}{\partial z} =$

$\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Sol

Given $\varphi = xy + yz + zx$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left[y+z\right] + \vec{j}\left[x+z\right] + \vec{k}\left[y+x\right]$$

At $(1, 2, 0)$

$$\nabla\varphi = 2\vec{i} + \vec{j} + 3\vec{k}$$

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3.$$

$$D.D. = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|} = \frac{(2\vec{i} + \vec{j} + 3\vec{k}) \cdot (\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \frac{2+2+6}{3} = \frac{10}{3}.$$

$$\text{Max. } D.D. = |\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}.$$

If $\vec{F} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$, find $\nabla \cdot \vec{F}$ at $(1, 2, -1)$

Sol

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

Sol

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x$$

$$= 0 + 0 + 0 = 0.$$

$\Rightarrow$ $\vec{F}$ is solenoidal.

(4) Find the directional derivative of $\varphi = 3x^2$ at $(1,1,1)$ in the direction $2\vec{i} + 2\vec{j} - \vec{k}$

Sol

Given $\varphi = 3x^2 + 2y - 3z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2+2y-3z) + \vec{j}\frac{\partial}{\partial y}(3x^2+2y-3z) + \vec{k}\frac{\partial}{\partial z}(3x^2+2y-3z)$$

$$= \vec{i}(6x) + \vec{j}(2) + \vec{k}(-3)$$

$$\nabla\varphi = 6x\vec{i} + 2\vec{j} - 3\vec{k}$$

At $(1,1,1)$

$$\nabla\varphi = 6\vec{i} + 2\vec{j} - 3\vec{k}$$

Solenoidal.

Sol

Let $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$.

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z)$$

$$= 3 + a + 2 = 5+a$$

Given $\vec{F}$ is solenoidal $\Rightarrow \text{div } \vec{F} = 0$

$\Rightarrow 5+a=0$

$\Rightarrow \boxed{a=-5}$

What is the greatest rate of increase of $\varphi = xyz^2$ at $(1,0,3)$.

Sol

Given $\varphi = xyz^2$.

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\, yz^2 + \vec{j}\, xz^2 + \vec{k}(2xyz)$$

At $(1,0,3)$

Let $\varphi = x^2+y^2+z^2-9$ $\qquad\psi = x^2+y^2-z$

$\nabla\varphi = \vec{i}\dfrac{\partial\varphi}{\partial x}+\vec{j}\dfrac{\partial\varphi}{\partial y}+\vec{k}\dfrac{\partial\varphi}{\partial z}$ $\qquad \nabla\psi = \vec{i}\dfrac{\partial\psi}{\partial x}+\vec{j}\dfrac{\partial\psi}{\partial y}$

$= 2x\vec{i}+2y\vec{j}+2z\vec{k}$ $\qquad \nabla\psi = 2x\vec{i}+2y\vec{j}$

At $(2,-1,2)$ $\qquad$ At $(2,-1,2)$

$\nabla\varphi = 4\vec{i}-2\vec{j}+4\vec{k}$ $\qquad \nabla\psi = 4\vec{i}-2\vec{j}$

$|\nabla\varphi| = \sqrt{16+4+16} = \sqrt{36}=6$ $\qquad |\nabla\psi| = \sqrt{16+4+}$

$$\cos\theta = \frac{\nabla\varphi\cdot\nabla\psi}{|\nabla\varphi|\,|\nabla\psi|} = \frac{16+4\cancel{+}4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

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(8) Prove that the vector $\vec{F}=(x+3y)\vec{i}+(y-3z)\vec{j}+(x-2z)\vec{k}$ is solenoidal.

proof

Given $\vec{F}=(x+3y)\vec{i}+(y-3z)\vec{j}+(x-2z)\vec{k}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-3z)+\frac{\partial}{\partial z}(x-2z)$$

Sol

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}\left[-2yz - 0\right] - \vec{j}\left[z^2 - xy\right] + \vec{k}\left[6xy - xz\right]$$

At $(1, 2, -1)$

$$\nabla \times \vec{F} = 4\vec{i} + \vec{j} + 13\vec{k}$$

Show that vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}\left[-1 + 1\right] - \vec{j}\left[3z^2 - 3z^2\right] + \vec{k}\left[6x - 6x\right]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

Sol

Let $\varphi = x^2 - y^2 - z^2 - 11$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= 2x\,\vec{i} - 2y\,\vec{j} - 2z\,\vec{k}$$

At (6,4,3)

$$\nabla\varphi = 12\,\vec{i} - 8\,\vec{j} - 6\vec{k}$$

$$|\nabla\varphi| = \sqrt{144+64+36}$$

$$= \sqrt{244}$$

$\psi = xy + yz - zx -$

$$\nabla\psi = \vec{i}\,\frac{\partial\psi}{\partial x} + \vec{j}\,\frac{\partial\psi}{\partial y} + \vec{k}$$

$$\nabla\psi = \vec{i}\,[\,y - z\,] + \vec{j}\,[\,x +$$

$$\vec{k}\,[\,y$$

$$\nabla\psi = \vec{i} + 9\,\vec{j} - 2\vec{k}$$

$$|\nabla\psi| = \sqrt{1+81+4}$$

$$\cos\theta = \frac{\nabla\varphi\cdot\nabla\psi}{|\nabla\varphi|\,|\nabla\psi|} = \frac{12 - 72 + 12}{\sqrt{244}\,\sqrt{86}} = \frac{-48}{\sqrt{244}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left[\frac{-48}{\sqrt{244}\,\sqrt{86}}\right]$$

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix}$$

$$= \vec{i}\left[-1+1\right] - \vec{j}\left[4xz - 4xz\right] + \vec{k}\left[2y - 2y\right]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$$

$\Rightarrow$ $\vec{F}$ is irrotational.

Let $\vec{F} = \nabla\varphi$

$$(y^2+2xz^2)\vec{i} + (2xy-z)\vec{j} + (2x^2z-y+2z)\vec{k} = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$$

$\frac{\partial\varphi}{\partial x} = y^2+2xz^2$ $\qquad$ $\frac{\partial\varphi}{\partial y} = 2xy-z$ $\qquad$ $\frac{\partial\varphi}{\partial z} = 2x^2z-y+2z$

$\varphi = xy^2 + x^2z^2 + f(y,z)$ $\qquad$ $\varphi = xy^2 - yz + g(x,z)$ $\qquad$ $\varphi = x^2z^2 - yz + z^2 + h(x,y)$

$$\varphi = xy^2 + x^2z^2 - yz + z^2 + c.$$

Show that $\vec{F} = (x^2-y^2+x)\vec{i} - (2xy+y)\vec{j}$ is irrotational. Find its scalar potential.

$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}$

$\Rightarrow \vec{F}$ is irrotational.

Let $\vec{F} = \nabla\varphi$

$\frac{\partial\varphi}{\partial x} = x^2 - y^2 + x$ $\qquad \frac{\partial\varphi}{\partial y} = -2xy - y$ $\qquad \frac{\partial\varphi}{\partial z} = 0$

$\varphi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + f(y,z)$ $\qquad \varphi = -xy^2 - \frac{y^2}{2} + g(x,z)$ $\qquad \varphi = h($

$$\varphi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c.$$

Show that $\vec{F} = (z^2 + 2x + 3y)\vec{i} + (3x + 2y + z)\vec{j} + (y + 2z$

is irrotational. Also find its scalar p

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z^2 + 2x + 3y & 3x + 2y + z \end{vmatrix}$$

$$= \vec{i}\left[1 - 1\right] - \vec{j}\left[2z - 2z\right] + \vec{k}\left[3 - 3\right]$$

$\phi = z^2x + x^2 + 3yx + f(y,z)$	$\phi = 3xy + y^2 + zy + g(x,z)$	$\phi = yz + z^2x + h(x,y)$

$$\phi = z^2x + x^2 + 3yx + y^2 + zy + c$$

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, P.T. $\text{Curl}\,(r^n\vec{r}) = \vec{0}$.

<u>proof</u>

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}\ ;\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ;\ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$r^n\vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\text{Curl}\,(r^n\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\left[n r^{n-1}\frac{\partial r}{\partial y} z - n r^{n-1}\frac{\partial r}{\partial z}\cdot y \right]$$

the vector

$$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$$

is irrotational. 2

Sol

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i}\left[c+1\right] - \vec{j}\left[4-a\right] + \vec{k}\left[b-2\right]$$

$\vec{F}$ is irrotational, $\Rightarrow$ Curl $\vec{F} = \vec{0}$

$c+1=0$ $\Rightarrow c=-1$

$4-a=0$ $a=4$

$b-2=0$ $b=2$

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{Curl } \vec{F} = \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\text{div}(\text{curl } \vec{F}) = \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$= 0.$

(2) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ & $r = |\vec{r}|$. Prove that $r^n\vec{r}$ is solenoidal if $n = -3$ &

$r^n \vec{r}$

$r^n \vec{r} = r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k}$.

$\operatorname{div}(r^n \vec{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= r^n . 1 + x . n r^{n-1} \frac{\partial r}{\partial x} + r^n . 1 + y . n r^{n-1} \frac{\partial r}{\partial y} +$

$r^n . 1 + z . n r^{n-1} \frac{\partial r}{\partial z}$

$= r^n + x n r^{n-1} . \frac{x}{r} + r^n + y n r^{n-1} . \frac{y}{r} +$

$r^n + z n r^{n-1} \frac{z}{r}$

$= r^n + n r^{n-2} x^2 + r^n + n r^{n-2} y^2 + r^n + n r^{n-2} z^2$

$= 3r^n + n r^{n-2} [x^2 + y^2 + z^2]$

$= 3r^n + n r^{n-2} . r^2$

$\operatorname{div}(r^n \vec{r}) = 3r^n + n r^n = (3+n) r^n$.

$r^n \vec{r}$ is solenoidal $\Rightarrow \operatorname{div}(r^n \vec{r}) = 0$

$\Rightarrow (3+n) r^n = 0$

$\Rightarrow \boxed{n = -3}$

$$\vec{j}\left[\frac{\partial(r^n z)}{\partial x} - \frac{\partial(r^n x)}{\partial z}\right] + \vec{k}\left[\frac{\partial(r^n y)}{\partial x} - \frac{\partial(r^n x)}{\partial y}\right]$$

$$= \vec{i}\left[z n r^{n-1}\frac{\partial r}{\partial y} - y n r^{n-1}\frac{\partial r}{\partial z}\right] -$$

$$\vec{j}\left[z n r^{n-1}\frac{\partial r}{\partial x} - x n r^{n-1}\frac{\partial r}{\partial z}\right] +$$

$$\vec{k}\left[y n r^{n-1}\frac{\partial r}{\partial x} - n r^{n-1} x \frac{\partial r}{\partial y}\right]$$

$$= \vec{i}\left[z n r^{n-1}\cdot\frac{y}{r} - y n r^{n-1}\frac{z}{r}\right]$$

$$-\vec{j}\left[z n r^{n-1}\frac{x}{r} - x n r^{n-1}\frac{z}{r}\right] +$$

$$\vec{k}\left[y n r^{n-1}\frac{x}{r} - n r^{n-1} x\cdot\frac{y}{r}\right]$$

$$= \vec{i}\left[n r^{n-2} z y - n r^{n-2} y z\right] - \vec{j}\left[n r^{n-2} x z - n r^{n-2} x z\right]$$

$$+\vec{k}\left[n r^{n-2} x y - n r^{n-2} x y\right]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k} = \vec{0}.$$

$= 2x + 2y + 2z.$

$$R.H.S = \int_0^c \int_0^b \int_0^a 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^c \int_0^b \left[\frac{x^2}{2} + xy + xz \right]_0^a dy\, dz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az \right) dy\, dz$$

$$= 2\int_0^c \left(\frac{a^2}{2}y + \frac{ay^2}{2} + azy \right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right) dz$$

$$= 2\left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2} \right]_0^c$$

$$= 2\left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right]$$

$$R.H.S. = a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

	Surfaces	$\vec{n}$	$\vec{F}\cdot\vec{n}$	ds
S_1	OBFC $(x=0)$	$-\vec{i}$	$-x^2 = 0$	dydz

$$\iint \vec{F}\cdot\vec{n}\, ds = 0$$

On S_2

$$\iint \vec{F}\cdot\vec{n}\, ds = \int_0^c\int_0^b a^2\, dy\, dz$$

$$= a^2 \int_0^c b\, dz = a^2 bc$$

On S_3

$$\iint \vec{F}\cdot\vec{n}\, ds = 0.$$

On S_4

$$\iint \vec{F}\cdot\vec{n}\, ds = \int_0^c\int_0^a b^2\, dx\, dz$$

$$= b^2 a \int_0^c dz = ab^2c.$$

On S_5

$$\iint \vec{F}\cdot\vec{n}\, ds = 0$$

On S_6.

$$\iint \vec{F}\cdot\vec{n}\, ds = \int^b\int^a c^2\, dx\, dy$$

by $x=0;\ y=0;\ z=0;\ x=1;\ y=1\ \&\ z=1.$

Sol

$$\iint \vec{F}\cdot\vec{n}\,ds = \iiint \operatorname{div}\vec{F}\,dv$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

$$\iiint \operatorname{div}\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[4zx - xy\right]_{x=0}^{1} dy\,dz$$

$$= \int_0^1\int_0^1 (4z-y)\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[\frac{\cancel{4}^{2}z^2}{\cancel{2}} - \frac{z}{2}\right]_0^1 = \left[2z^2 - \frac{z}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

R.H.S $= \frac{3}{2}$.

$\uparrow z$

			$-4xz = 0$	$dy\,d$
S_2	$x=1$	$\vec{i}$	$4xz = 4z$	$dy\,d$
S_3	$y=0$	$-\vec{j}$	$y^2 = 0$	$dx\,d$
S_4	$y=1$	$\vec{j}$	$-y^2 = -1$	$dx\,d$
S_5	$z=0$	$-\vec{k}$	$-yz = 0$	$dx\,d$
S_6	$z=1$	$\vec{k}$	$yz = y$	$dx\,d$

On S_1

$$\iint \vec{F}\cdot\vec{n}\,ds = 0$$

On S_2

$$\iint \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1 4z\,dy\,dz$$

$$= \int_0^1 4z\,dz = \left[\frac{4z^2}{2}\right]_0^1 = \frac{4}{2} = 2.$$

On S_3

$$\iint \vec{F}\cdot\vec{n}\,ds = 0$$

On S_4

$$\iint \vec{F}\cdot\vec{n}\,ds = \int_0^1\int_0^1 -1\,dx\,dz = -1$$

① Verify Stoke's theorem for $\vec{F} = (y-z)\vec{i} + yz\vec{j} - xz\vec{k}$ where S is the Surface bounded by the planes $x=0$; $x=1$; $y=0$; $y=1$ & $z=1$ above XY - plane

Sol

By Stoke's theorem

$$\int \vec{F}\cdot d\vec{r} = \iint \text{curl}\vec{F}\cdot\vec{n}\, ds.$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix}$$

$$= \vec{i}\left[0 - y\right] - \vec{j}\left[-z+1\right] + \vec{k}\left[0-1\right]$$

$$\text{Curl}\,\vec{F} = -y\vec{i} - (1-z)\vec{j} - \vec{k}$$

Surfaces	$\vec{n}$	Curl $\vec{F}\cdot\vec{n}$	ds
S1 : $x=0$	$-\vec{i}$	y	dydz
S2 : $x=1$	$\vec{i}$	$-y$	dydz

$$\iint \text{Curl } F \cdot n \; ds = \int_0\int_0 y \, dy \, dz$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dz = \int_0^1 \frac{1}{2} dz = \left[\frac{z}{2}\right]_0^1$$

$$= \frac{1}{2}$$

On S_2

$$\iint \text{Curl } \vec{F} \cdot \vec{n} \; ds = \int_0^1\int_0^1 -y \; dy \, dz = -\int_0^1\int_0^1 y \, dy \, dz = \frac{-1}{2}$$

On S_3

$$\iint \text{curl } \vec{F} \cdot \vec{n} \; ds = \int_0^1\int_0^1 (1-z) \; dx \, dz$$

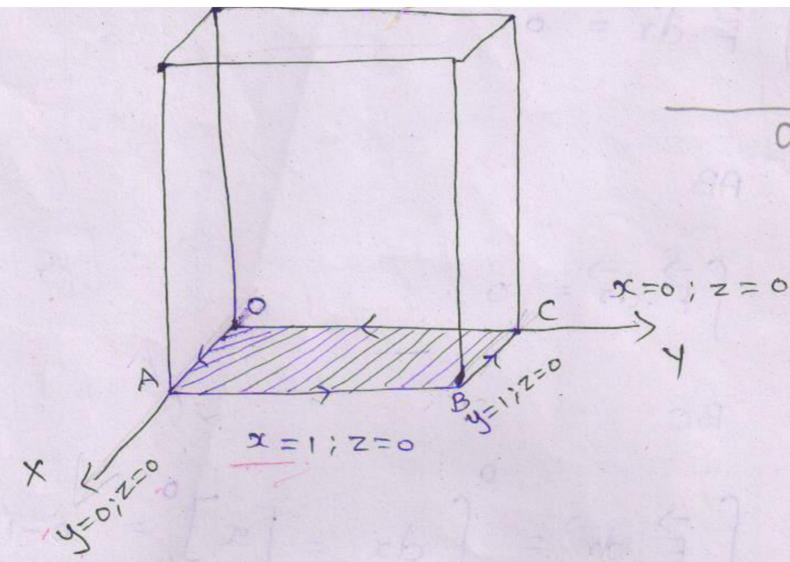
$$= \int_0^1 \left[x - xz\right]_0^1 dz$$

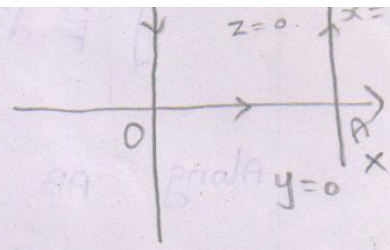
$$= \int_0^1 (1-z) \, dz = \left[z - \frac{z^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

On S_4

$$\iint \text{Curl } \vec{F} \cdot \vec{n} \; ds = \int_0^1\int_0^1 (z-1) \; dx \, dz$$

$$= \int_0^1 \left[zx - x\right]_0^1 dz$$





~~Curves~~ $\vec{F} = (y-z)\vec{i} + yz\,\vec{j} - xz\,\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ ← by th

$\vec{F}\cdot d\vec{r} = (y-z)dx + yz\,dy - xz\,dz$

Curves		Limits	$\vec{F}\cdot d\vec{r}$
OA	$y=0\ ;\ z=0$ $dy=0 \quad dz=0$	$x: 0$ to 1	0
AB	$x=1\ ;\ z=0$ $dx=0 \quad dz=0$	$y: 0$ to 1	0
BC	$y=1\ ;\ z=0$ $dy=0 \quad dz=0$	$x: 1$ to 0	dx

$\int \vec{F}\cdot d\vec{r} = 0$

Along AB

$$\int \vec{F}\cdot d\vec{r} = 0$$

Along BC

$$\int \vec{F}\cdot d\vec{r} = \int_1^0 dx = [x]_1^0 = 0-1 = -1.$$

Along CO

$$\int \vec{F}\cdot d\vec{r} = \int 0 = 0.$$

L.H.S = 0+0-1+0 = -1.

$\therefore$ L.H.S = R.H.S.

Stoke's thm is verified.

Verify Stoke's thm for $\vec{F} = xy\,\vec{i} - 2yz\,\vec{j} - zx\,\vec{k}$ where S is the open surface of the rectangular parallelopiped formed by the planes $x=0$; $x=1$; $y=0$; $y=2$ & $z=3$ above

$$\begin{vmatrix} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \\ & xy & -2yz & -zx \end{vmatrix}$$

$$= \vec{i}\,[0 + 2y] - \vec{j}\,[-z - 0] + \vec{k}\,[0 - x]$$

$$= 2y\,\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Surfaces		$\vec{n}$	$Curl\,\vec{F}\cdot\vec{n}$	ds
S1	$x = 0$	$-\vec{i}$	$-2y$	$dy\,dz$
S2	$x = 1$	$\vec{i}$	$2y$	$dy\,dz$
S3	$y = 0$	$-\vec{j}$	$-z$	$dx\,dz$
S4	$y = 2$	$\vec{j}$	z	$dx\,dz$
S5	$z = 3$	$\vec{k}$	$-x$	$dx\,dy$

On S_1

$$\iint Curl\,\vec{F}\cdot\vec{n}\,ds = \int_0^3\int_0^2 -2y\;dy\,dz$$

$$= \int_0^3 \left[\frac{-\cancel{2}y^2}{\cancel{2}}\right]_0^2 dz$$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

$\vec{F}\cdot d\vec{r} = xy\,dx - 2yz\,dy - zx\,dz$

~~Curves~~

	Curves	Limits	$\vec{F}\cdot d\vec{r}$
OA:	$y=0$; $z=0$ $dy=0 \quad dz=0$	$x : 0$ to 1	0
AB	$x=1$; $z=0$ $dx=0 \quad dz=0$	$y : 0$ to 2	0
BC	$y=2$; $z=0$ $dy=0 \quad dz=0$	$x : 1$ to 0	$2x\,dx$
CO	$x=0$; $z=0$ $dx=0 \quad dz=0$	$y : 2$ to 0.	0

Along OA

$$\int \vec{F}\cdot d\vec{r} = 0$$

Along ~~of~~ AB

$$\int \vec{F}\cdot d\vec{r} = 0.$$

Along BC

$$\int \vec{F}\cdot d\vec{r} = \int^{0} \ldots \left[2x^2\right]^0$$

$$\int M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

(1) Evaluate $\displaystyle\int (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the lines $x=\pm 1$; $y=\pm 1$

Sol

By Green's thm,

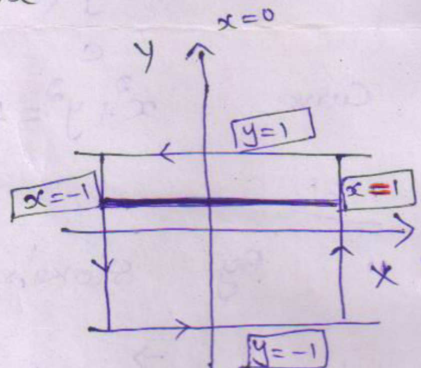
$$\int M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M\,dx + N\,dy = (x^2+xy)\,dx + (x^2+y^2)\,dy$

Here $M = x^2+xy$ $\qquad N = x^2+y^2$

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x.$$



$$\therefore \int (x^2+xy)\,dx + (x^2+y^2)\,dy$$

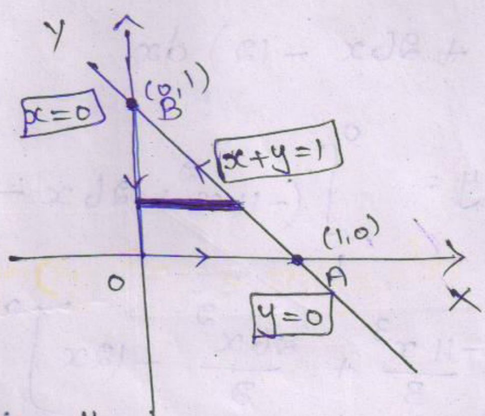
$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{-1}^{1} dy = \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\right) dy = 0.$$

$M = 3x^2 - 8y^2$ $\qquad$ $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\qquad$ $\frac{\partial N}{\partial x} = -6y$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$



$x + y = 1$

x	0	1
y	1	0

$(0,1), (1,0)$

$$R.H.S = \int_0^1 \int_0^{1-y} 10y \, dx \, dy$$

$$= \int_0^1 10y(1-y)\,dy = 10\int_0^1 (y - y^2)\,dy$$

$$= 10\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = 10\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{10}{6} = \frac{5}{3}.$$

Along OA ; $y = 0 \Rightarrow dy = 0$

$M\,dx + N\,dy = 3x^2\,dx.$

$\therefore \int M\,dx + N\,dy = \int^1 3x^2\,dx = \left[\frac{3x^3}{}\right]^1 = 1.$

$$= \left[3x^2 - 2(1+x^2-2x)\right]dx - \left[4-4x-6x+6x^2\right]d$$

$$= \left[3x^2 - 2 - 2x^2 + 16x\right]dx - \left[4 - 10x + 6x^2\right]dx$$

$$= \left[3x^2 - 2 - 2x^2 + 16x - 4 + 10x - 6x^2\right]dx$$

$$= (-11x^2 + 26x - 12)\,dx.$$

$$\therefore \int M\,dx + N\,dy = \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$$= \left[\frac{-11x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0$$

$$= 0 - \left(\frac{-11}{3} + \frac{26}{2} - 12\right)$$

$$= \frac{11}{3} - \frac{26}{2} + 12 = \frac{2}{3}.$$

Along BO

$$x = 0 \qquad dx = 0.$$

$$M\,dx + N\,dy = 4y\,dy$$

$$\int M\,dx + N\,dy = \int_1^0 4y\,dy = \left[\frac{4y^2}{2}\right]_1^0 = 0 - 2 = -2.$$

$$L.H.S = \left[1 + \frac{2}{3}\right] - 2 = \frac{5}{\ }$$

...oundary of the region ~~bounded~~ defined by $x = y^2$; $y = x^2$.

Sol

$$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$$

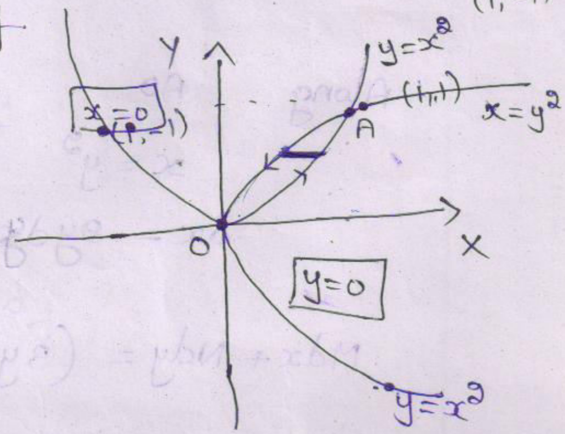
$$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$$

$x = y^2$

x	0	1
y	0	± 1

$(0,0), (1,1)$
$(1,-1)$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$



$$\text{R.H.S} = \iint 10y \, dx\, dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y \, dx\, dy$$

y	0	1
x	0	± 1

$(0,0), (1,1)$
$(-1,1)$

$$= \int_0^1 10y \left[\sqrt{y} - y^2\right] dy$$

$$= 10\int_0^1 \left(y^{3/2} - y^3\right) dy$$

$$= 10\left[\frac{2y^{5/2}}{5} - \frac{y^4}{4}\right]_0^1 = 10\left[\frac{2}{5} - \frac{1}{4}\right] = \cancel{\frac{2}{3}}\ \frac{3}{2}$$

$$= (3x^2 + 8x^3 - 20x^4)\,dx$$

$$\int M\,dx + N\,dy = \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1$$

$$= \frac{3}{3} + \frac{8}{4} - \frac{20}{5} = -1.$$

Along AO

$$x = y^2$$

$$dx = 2y\,dy$$

$$M\,dx + N\,dy = (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^2 \cdot y)\,dy$$

$$= (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= (6y^5 - 22y^3 + 4y)\,dy$$

$$\int M\,dx + N\,dy = \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right]_1^0$$

$$= 0 - \left[1 - \frac{22}{4} + 2\right] = +\frac{5}{2}.$$

Sol

The gradiant of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

2. Define the directional derivative

Sol

The derivative of a point function in a particular direction is called its directional derivative along the direction.

The directional derivative of φ along the direction of $\vec{a}$ is

$$\frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

3. Define divergence of a vector function

Sol

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$. Then

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$
$(\nabla\cdot\vec{F})$

4. Define Solenoidal vector

A vector $\vec{F}$ is said to be Solenoidal if $\text{div}\,\vec{F} = 0$.

5. Define curl of vector function

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$.

Then
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$(\nabla\times\vec{F})$

is said to be rotation of $\vec{F}$

... irrotational if $\text{curl}\,\vec{F} = 0$ i.e. $\nabla\times\vec{F} = \vec{0}$

7. Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2,-2,-1)$

Sol

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$
$$= \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$
$$(\nabla\varphi)_{(2,-2,-1)} = \vec{i}(2+8) - 4\vec{j} - 16\vec{k}$$
$$= 10\vec{i} - 4\vec{j} - 16\vec{k}$$
$$|\nabla\varphi| = \sqrt{10^2 + (-4)^2 + (-16)^2}$$
$$= \sqrt{100+16+256} = \sqrt{372}$$

8. Find the directional derivative of $\varphi = xy + yz + zx$ at $(1,2,0)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value

Sol

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$
$$= \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(y+x)$$
$$(\nabla\varphi)_{(1,2,0)} = 2\vec{i} + \vec{j} + 3\vec{k}$$
$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$
$$|\vec{a}| = \sqrt{1+2^2+2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$\therefore$ directional derivative $= \dfrac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$
$$= \frac{(2\vec{i}+\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$$
$$= \frac{2(1)+1(2)+3(2)}{3} = \frac{10}{3}$$

Maximum value $= |\nabla\varphi| = \sqrt{2^2+1^2+3^2}$
$= \sqrt{14}$

Sol

Given $\vec{F} = (x+2y)\vec{i} + (y+3z)\vec{j} + (x-2z)\vec{k}$

Here $F_1 = x+2y$; $F_2 = y+3z$; $F_3 = x-2z$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1+1-2 = 2-2 = 0$$

$\therefore \vec{F}$ is solenoidal.

10\. If $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, show that $\operatorname{curl}\operatorname{curl}\vec{F} = 0$

Sol

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \vec{i}\cdot 1 - \vec{j}(0-1) + \vec{k}(1-0)$$

$$\operatorname{curl}\vec{F} = \vec{i} + \vec{j} + \vec{k}$$

$$\operatorname{curl}\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$\operatorname{curl}\operatorname{curl}\vec{F} = \vec{0}$$

11\. Prove that $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$.

proof

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Then
$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\ldots\left[\frac{\partial \ldots}{\partial x} - \frac{\partial \ldots}{\partial z}\right] \ldots \frac{\partial}{\partial z}\left[\frac{\partial \ldots}{\partial x} - \frac{\partial \ldots}{\partial y}\right]$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}} = 0.$$

12\. Define Conservative force field.

If the workdone by the force $\vec{F}$ along $C = \int_C \vec{F}\cdot d\vec{r}$ is independent of the path joining the two points then $\vec{F}$ is said to be a conservative force field.

13\. State Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where

$M, N \rightarrow$ Continuous functions with Continuous partial derivatives

$C \rightarrow$ closed curve

$R \rightarrow$ Region enclosed by C.

14\. State Stoke's thm.

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$\vec{n} \rightarrow$ unit normal vector to S

$C \rightarrow$ closed curve.

15\. State Gauss divergence theorem

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$V \rightarrow$ volume bounded by a closed surface S.

$\vec{F} \rightarrow$ Continuous and has continuous partial ...