UNIT 4 – ANALYTIC FUNCTIONS

Let $z = x+iy$ be a complex variable

Then $\omega = f(z) = u+iv$ be the complex valued function of complex variable.

Analytic Function (Regular or Holomorphic Function)

A function $\omega = f(z)$ defined at z_0 is analytic at z_0 if it has a derivative at z_0 and at every point in some neighbourhood of z_0.

Necessary Condition for $f(z)$ to be analytic

1) u_x ; v_x ; u_y ; v_y exist.

2) $u_x = v_y$; $v_x = -u_y$ (C–R equation).

Sufficient condition for $f(z)$ to be analytic

1) u_x, v_x, u_y, v_y are continuous

① Is the function $f(z) = \bar{z}$ analytic ?

Sol

$$f(z) = \bar{z}$$

$$\Rightarrow \quad u + iv = \overline{x+iy} = x - iy$$

$u = x$ $\qquad\qquad$ $v = -y$

$u_x = 1$ $\qquad\qquad$ $v_x = 0$

$u_y = 0$ $\qquad\qquad$ $v_y = -1$

Here $u_x \neq v_y \Rightarrow$ CR equation is not satisfied.

$\Rightarrow$ $f(z) = \bar{z}$ is analytic.

② Verify $f(z) = z^3$ is analytic or not.

Sol

$$f(z) = z^3$$

$$u + iv = (x+iy)^3 = x^3 - iy^3 + 3x^2 iy - 3xy^2$$

$\Rightarrow$ $u = x^3 - 3xy^2$ $\qquad\qquad$ $v = 3x^2 y - y^3$

$u_x = 3x^2 - 3y^2$ $\qquad\qquad$ $v_x = 6xy$

③ Show that $|z|^2$ is not analytic at any point.

Sol

$$f(z) = |z|^2$$

$$u+iv = |x+iy|^2 = x^2+y^2$$

$$u = x^2+y^2 \qquad v = 0.$$

$u_x = 2x$ | At $(0,0)$ $u_x = 0$ | $v_x = 0$

$u_y = 2y$ | $u_y = 0.$ | $v_y = 0.$

At $(0,0)$, $\left.\begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}\right\} \Rightarrow |z|^2$ is analytic at $(0,0)$.

From this, $|z|^2$ is not analytic at any point.

④ Prove that $w = \sin(2z)$ is an analytic function.

Proof

$u = \sin(2x)\cosh(2y)$ $v = \cos(2x)\sinh(2y)$

$u_x = 2\cos(2x)\cosh(2y)$ $v_x = -2\sin(2x)\sinh(2y)$

$u_y = 2\sin(2x)\sinh(2y)$ $v_y = 2\cos(2x)\cosh(2y)$

$u_x = v_y$; $v_x = -u_y$

$\Rightarrow$ CR equ. is Satisfied.

$\Rightarrow$ $w = \sin(2z)$ is analytic.

(5) Test the analyticity of $f(z) = e^z$.

Sol

$f(z) = e^z$

$u + iv = e^{x+iy} = e^x[\cos y + i\sin y]$

$u = e^x \cos y$ $v = e^x \sin y$

$u_x = e^x \cos y$ $v_x = e^x \sin y$

$u_y = -e^x \sin y$ $v_y = e^x \cos y$.

(6) Test the analyticity of $f(z) = z^n$.

Sol)

$$f(z) = z^n.$$

$$u+iv = (re^{i\theta})^n = r^n \cdot e^{in\theta}$$

$$u+iv = r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$u = r^n \cos(n\theta) \qquad v = r^n \sin(n\theta)$$

$$\frac{\partial u}{\partial r} = n \cdot r^{n-1} \cos(n\theta) \qquad \frac{\partial v}{\partial r} = n \cdot r^{n-1} \sin(n\theta)$$

$$\frac{\partial u}{\partial \theta} = -n \cdot r^n \sin(n\theta) \qquad \frac{\partial v}{\partial \theta} = n \cdot r^n \cos(n\theta)$$

From this, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r} \cdot \dfrac{\partial v}{\partial \theta}$

$$\frac{\partial v}{\partial r} = \frac{-1}{r} \cdot \frac{\partial v}{\partial \theta}$$

$\Rightarrow$ Cauchy – Riemann equation is satisfied

$\Rightarrow$ $f(z) = z^n$ is analytic.

(7) Find the constants a, b, c, if

$u = x + ay$ $\qquad$ $v = bx + cy$

$u_x = 1$ $\qquad$ $v_x = b$

$u_y = a$ $\qquad$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow$ $u_x = v_y$ $\Rightarrow$ $\boxed{1 = c}$

$v_x = -u_y$ $\Rightarrow$ $\boxed{b = -a}$

(8) Find the constants a, b if $f(z) = x + 2ay + i(3x + by)$ is analytic.

Sol

Given $f(z) = x + 2ay + i(3x + by)$

$u + iv = x + 2ay + i(3x + by)$

$u = x + 2ay$ $\qquad$ $v = 3x + by$

$u_x = 1$ $\qquad$ $v_x = 3$

$u_y = 2a$ $\qquad$ $v_y = b$.

Harmonic Function

A function $u(x,y)$ is called harmonic when $u_{xx} + u_{yy} = 0$.

① Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

Verification

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$u_x = 3x^2 - 3y^2 + 6x$ $\qquad$ $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$ $\qquad$ $u_{yy} = -6x - 6$

$u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

$\Rightarrow$ u is harmonic.

② Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

Sol

$$u = 2x - x^3 + 3xy^2$$

Properties of Analytic Function

① Prove that the real and imaginary parts of an analytic function are harmonic.

proof

Take $f(z) = u+iv$ be analytic.

Then $u_x = v_y$; $v_x = -u_y$
$\quad\hookrightarrow ①$ $\quad\hookrightarrow ②$

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Diff ① w.r.t. x, $u_{xx} = v_{xy}$

Diff ② w.r.t. y, $v_{yx} = -u_{yy}$

$\Rightarrow u_{yy} = -v_{yx}$

$\therefore u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$

$\Rightarrow$ u is harmonic.

Similarly, v is harmonic.

Note

From the above property,

Proof

$u = x^2 - y^2$

$u_x = 2x \qquad u_y = -2y$

$u_{xx} = 2 \qquad u_{yy} = -2$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic.

$$v = \frac{-y}{x^2+y^2} = -y\,(x^2+y^2)^{-1}$$

$$v_x = y\,(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)\left[\cancel{2x^2y + 2y^3} - x^2 + y^2 - 4x^2\right]}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{2y\,(y^2 - 3x^2)}{(x^2+y^2)^3}$$

$$v_y = \frac{-(x^2+y^2) + y\,(2y)}{\ldots} = \frac{y^2 - x^2}{\ldots}$$

$$= \frac{2y(x^2+y^2)\left[x^2+y^2-2y^2+2x^2\right]}{(x^2+y^2)^4}$$

$$V_{yy} = \frac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$$

$V_{xx} + V_{yy} = 0 \Rightarrow V$ is harmonic.

$U_x \neq V_y \Rightarrow$ C-R equations are not satisfied.

$\Rightarrow f(z) = u+iv$ is not analytic.

(2) Prove that an analytic function with constant real part (Imaginary part) is constant.

proof

Let $f(z) = u+iv$ be analytic function.

Then $U_x = V_y$; $V_x = -U_y$.

Take $u =$ Constant

(3) Prove that an analytic function with Constant modulus is constant.

proof

Let $f(z) = u + iv$ be analytic.

Then $u_x = v_y$; $v_x = -u_y$.

$|f(z)| = \sqrt{u^2+v^2}$.

Given Modulus = Constant

$\Rightarrow \sqrt{u^2+v^2} = c$ (1)

$\Rightarrow u^2+v^2 = c^2$.

Diff. w.r.t. x

$2uu_x + 2vv_x = 0$

$uu_x + vv_x = 0$ $\rightarrow$ (1)

Diff. w.r.t. y

$2uu_y + 2vv_y = 0$.

$-uv_x + vu_x = 0$. $\rightarrow$ (2)

(1) $\times v$ $\Rightarrow$ $uv\,u_x + v^2 v_x = 0$.

(4) If $f(z) = u+iv$ is analytic, prove that the curves $u =$ Constant & $v =$ Constant are orthogonal.

Proof

Given $f(z) = u+iv$ is analytic

$\Rightarrow \quad u_x = v_y \quad ; \quad v_x = -u_y$

Given $u(x,y) =$ Constant — (1)

Slope of (1) is $\boxed{m_1 = \frac{-u_x}{u_y}}$

$v(x,y) =$ Constant — (2)

Slope of (2) is $m_2 = \frac{-v_x}{v_y}$

$m_1 m_2 = -\frac{u_x}{u_y} \times \frac{-v_x}{v_y} = \frac{u_x}{u_y} \times \frac{-u_y}{u_x} = -1$

$\Rightarrow$ The curves are orthogonal.

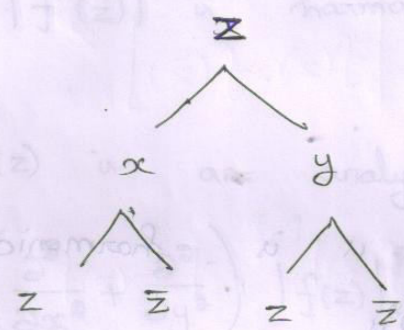
Result

$\frac{\partial^2}{} + \frac{\partial^2}{} \quad \partial^2$

$x = \dfrac{z+\bar{z}}{2}$ $\qquad$ $y = \dfrac{z-\bar{z}}{2i}$

$\dfrac{\partial x}{\partial z} = \dfrac{1}{2}$ $\qquad$ $\dfrac{\partial y}{\partial z} = \dfrac{1}{2i}$

$\dfrac{\partial x}{\partial \bar{z}} = \dfrac{1}{2}$ $\qquad$ $\dfrac{\partial y}{\partial \bar{z}} = \dfrac{-1}{2i}$



$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\boxed{\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}}$$

$$\boxed{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial z\partial\bar{z}}}$$

(17) Show that a harmonic function u Satisfies the formal differential equation $\frac{\partial^2 u}{\partial z\partial\bar{z}}=0$.

proof

Given u is harmonic.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

WKT $4\frac{\partial^2}{\partial z\partial\bar{z}}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$

$$\Rightarrow \frac{\partial^2 u}{\partial z\partial\bar{z}}=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0.$$

If $f(z)$ is a regular function, P.T. $\log|f'(z)|$ is harmonic.

proof

$$= 4 \left\{ \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ \log f(z) + \log f'(\bar{z}) \right\} \right\}$$

$$= 4 \left\{ \frac{\partial}{\partial z} \left[\frac{f''(\bar{z})}{f'(\bar{z})} \right] \right\} = 0.$$

$\Rightarrow \log |f'(z)|$ is harmonic.

(2) If $f(z)$ is an analytic function, Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}$.

Proof

WKT $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ |f(z)|^2 \right\}^{p/2}$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ f(z) \cdot f(\bar{z}) \right\}^{p/2}$$

$$= 4 \frac{\partial^2}{\partial z} \left\{ f(z)^{p/2} \cdot (p/2)\, f(\bar{z})^{p/2 - 1} \cdot f'(\bar{z}) \right\}$$

$$= 4 \left\{ \frac{p}{2} \left[f(z)^{p/2 - 1} \cdot f'(z) \right] \cdot \frac{p}{2} \cdot f(\bar{z})^{p/2 - 1} \cdot f'(\bar{z}) \right\}$$

$$= \frac{4 \times p^2}{4} \left[f(z) \cdot f(\bar{z}) \right]^{\frac{p-2}{2}} \cdot \left[f'(z) \cdot \overline{f'(z)} \right]$$

$$= p^2 \, |f(z)|^{p-2} \cdot |f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 \, |f(z)|^{p-2} \cdot |f'(z)|^2 .$$

When $p = 2$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 \, |f'(z)|^2 .$$

Construction of Analytic Functions

Milne - Thompson Method

① Construct an analytic function $f(z)$ for which the real part is $e^x \cos y$.

Sol

$$u = e^x \cos y$$

* $u_x = e^x \cos y$

$u_y = -e^x \sin y$

* Put $x = z$; $y = 0$

$u_x = e^z$

$u_y = 0$.

* $f(z) = \int (u_x - i u_y)\, dz + c$

$= \int e^z\, dz + c'$

$\Rightarrow \boxed{f(z) = e^z + c}$

② Prove that $u = 2x - x^3 + 3xy^2$ is harmonic. Determine its harmonic conjugate.

$u_{xx} + u_{yy} = 0$.

$\Rightarrow$ u is harmonic.

Put $x = z$; $y = 0$ in u_x & u_y

$u_x = 2 - 3z^2$

$u_y = 0$

$f(z) = \int (u_x - i u_y)\, dz + c$

$= \int (2 - 3z^2)\, dz + c$

$= 2z - \frac{3z^3}{3} + c$

$\boxed{f(z) = 2z - z^3 + c}$

$u + iv = 2(x+iy) - (x+iy)^3 + c$

$= 2x + 2iy - x^3 + iy^3 - 3x^2 iy$

$+ 3xy^2 + c_1 + i c_2$

$\Rightarrow u = 2x - x^3 + 3xy^2 + c_1$

(3) Find the analytic function $u+iv$ if $u = (x-y)(x^2+4xy+y^2)$. Also find the Conjugate harmonic function v.

Sol)

$$u = (x-y)(x^2+4xy+y^2)$$

$$u = x^3+4x^2y+xy^2-x^2y-4xy^2-y^3$$

* $u_x = 3x^2+8xy+y^2-2xy-4y^2$

$u_y = 4x^2+2xy-x^2-8xy-3y^2$

* Put $x = z$; $y = 0$

$u_x = 3z^2$ $\qquad u_y = 4z^2 - z^2 = 3z^2$

$\therefore f(z) = \int (u_x - i u_y)\, dz + c$

$$= \int (3z^2 - i\,3z^2)\, dz + c$$

$$= 3(1-i) \int z^2\, dz + c$$

$$u+iv = (1-i)(x+iy)^3 + c$$

$$= (1-i)\left[x^3 - iy^3 + 3x^2 iy - 3xy^2\right] + c$$

$$u+iv = x^3 - iy^3 + 3x^2 iy - 3xy^2 - ix^3$$

$$-y^3 + 3x^2 y + ixy^2 + c_1 + ic_2$$

$$\therefore\ u = x^3 - 3xy^2 + 3x^2y - y^3 + c_1$$

$$v = -y^3 + 3x^2 y - x^3 + xy^2 + c_2.$$

(4) Show that $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Determine its analytic function. Also find its conjugate.

Sol

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$u_x = \frac{1}{2} \times \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad (2)$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic

Put $x = z$; $y = 0$.

$$\therefore u_x = \frac{z}{z^2} = \frac{1}{z}$$

$$u_y = 0.$$

$$f(z) = \int (u_x - i u_y)\, dz + c'$$

$$= \int \frac{1}{z}\, dz + c'$$

$$\boxed{f(z) = \log z + c'}$$

$$u + iv = \log\left[r \cdot e^{i\theta}\right] + c'$$

$$= \log r + \log e^{i\theta} + c'$$

$$= \log r + i\theta + c'$$

(5) P.T. $u = e^x[x\cos y - y\sin y]$ is harmonic and hence find the analytic function $u+iv$.

Sol

$$u = e^x[x\cos y - y\sin y]$$

$$u_x = e^x[\cos y] + e^x[x\cos y - y\sin y]$$

$$u_{xx} = e^x\cos y + e^x[\cos y] + e^x[x\cos y - y\sin y]$$

$$u_y = e^x[-x\sin y - y\cos y - \sin y]$$

$$u_{yy} = e^x[-x\cos y + y\sin y - \cos y - \cos y]$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic.

Put $x = z$; $y = 0$ in u_x & u_y

$$\therefore \; u_x = e^z + z \cdot e^z = (1+z)\,e^z$$

$$u_y = e^z[\;\;]$$

$$= \left[(1+z)\ e^{z} \qquad - e^{z}\right] + c$$

$$f(z) = z e^{z} + c$$

(6) Given that $u = \dfrac{\sin(2x)}{\cosh(2y) - \cos(2x)}$

Find the analytic function $f(z) = u + iv$.

Sol

$$u = \frac{\sin(2x)}{\cosh(2y) - \cos(2x)}$$

$$u_x = \frac{2\left[\cosh(2y) - \cos(2x)\right]\cos(2x) - 2\sin^2(2x)}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

$$= \frac{2\cosh(2y)\cos(2x) - 2\cos^2(2x) - 2\sin^2(2x)}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

$$u_x = \frac{2\cosh(2y)\cos(2x) - 2}{\left[\cosh(2y) - \cos(2x)\right]^2}$$

Put $x = z$; $y = 0$.

$$u_x = \frac{2\cos(2z) - 2}{[1 - \cos(2z)]^2} = \frac{-2}{1 - \cos(2z)}$$

$$= \frac{-2}{2\sin^2(z)} = -\operatorname{cosec}^2 z$$

$u_y = 0$.

$$f(z) = \int (u_x - i u_y)\, dz + c$$

$$= -\int \operatorname{cosec}^2 z \, dz + c$$

$$f(z) = \cot z + c$$

(7) Prove that $e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function and the imaginary part.

Sol

$$u = e^{-2xy} \sin(x^2 - y^2)$$

$$u_x = 2x e^{-2xy} \cos(x^2 - y^2) - 2y e^{-2xy} \sin(x^2 - y^2)$$

$$f(z) = \int (u_x - i u_y)\, dz + c'$$

$$= \int \left[2z \cos(z^2) + 2iz \sin(z^2) \right] dz + c'$$

$$= 2\int z \left[\cos(z^2) + i \sin(z^2) \right] dz + c'$$

$$= 2\int z \cdot e^{iz^2} dz + c'$$

$$\cancel{= 2\left\{ z\frac{e^{iz^2}}{i2z} - 1\, e^{iz^2} \right\} + c'}$$

$$\cancel{= 2\times e^{iz^2}\left[\frac{1}{2i} - 1\right] + c'}$$

Take $t = z^2 \Rightarrow dt = 2z\,dz$

$$\Rightarrow z\,dz = \frac{dt}{2}$$

$$= \int e^{it}\, dt + c' = \frac{e^{it}}{i} + c'$$

$$\boxed{f(z) = -i\, e^{it} + c'} = -i e^{iz^2} + c'$$

$$u+iv = -i\, e^{-2xy}\left[\cos(x^2-y^2) + i\sin(x^2-y^2)\right] + c$$

$$\Rightarrow \quad v = -\left(e^{-2xy}\cos(x^2-y^2)\right)$$

Type (2) - Imaginary part v is given

(i) Find V_x and V_y

(ii) Put $x = z$ and $y = 0$.

(iii) $f(z) = \int (V_y + i V_x)\, dz + c$.

① Show that $v = e^{-x}\left[x\cos y + y\sin y\right]$ is harmonic function. Hence find its analytic function $f(z) = u+iv$.

Sol

$$V = e^{-x}\left[x\cos y + y\sin y\right]$$

$$V_x = e^{-x}\cos y - e^{-x}\left[x\cos y + y\sin y\right]$$

$$V_{xx} = -e^{-x}\cos y - e^{-x}\cos y + e^{-x}\left[x\cos y + y\sin y\right]$$

$$V_y = e^{-x}\left[-x\sin y + y\cos y + \sin y\right]$$

Put $x = z$; $y = 0$.

$$V_x = e^{-z} - e^{-z}.z = (1-z)e^{-z}$$

$$V_y = 0.$$

$$f(z) = \int (V_y + iV_x)\, dz + c$$

$$= \int (1-z)e^{-z}\, dz + c$$

$$= \left[(1-z)\frac{e^{-z}}{-1} + e^{-z}\right] + c$$

$$f(z) = -(1-z)e^{-z} + e^{-z} + c$$

(2) Can $v = \tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so Construct an analytic function $f(z) = u + iv$, taking v as the imaginary part and hence find u.

Sol

Condition to Imaginary part : v is harmonic.

$$V_{xx} = + y\,(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$V_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \times \frac{1}{x}$$

$$V_y = \frac{x}{x^2+y^2} = x\,(x^2+y^2)^{-1}$$

$$V_{yy} = -x\,(x^2+y^2)^{-2} \cdot 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$V_{xx} + V_{yy} = 0 \Rightarrow V$ is harmonic.

Put $x = z$; $y = 0$.

$$V_x = 0 \;;\quad V_y = \frac{z}{z^2+0} = \frac{z}{z^2} = \frac{1}{z}$$

$$f(z) = \int (V_y + iV_x)\,dz + c$$

$$= \int \frac{1}{z}\,dz + c$$

$$f(z) = \log z + c$$

Put, $z = re^{i\theta}$

$u = \log r = \frac{1}{2}\log(x^2+y^2)$

$\Rightarrow v = \theta = \tan^{-1}\left(\frac{y}{x}\right).$

③ If $f(z) = u+iv$ is an analytic function, find $f(z)$ if $v = \log(x^2+y^2) + x - 2y$.

Sol

$$v = \log(x^2+y^2) + x - 2y$$

$$V_x = \frac{2x}{x^2+y^2} + 1 \qquad V_y = \frac{2y}{x^2+y^2} - 2$$

Put $x = z$; $y = 0$

$$V_x = \frac{2z}{z^2} + 1 = \frac{2}{z} + 1 \qquad V_y = -2$$

$$f(z) = \int (V_y + iV_x)\, dz + c$$

$$= \int \left(-2 + \frac{i2}{z} + i\right) dz + c$$

$$f(z) = -2z + 2i\log z + iz + c$$

Type (3) — $u-v$ is given

* $f(z) = \dfrac{F(z)}{1+i}$

~~Method 4~~

Type (4) — $u+v$ is given

Take $V = u+v$

* Find V_x and V_y

* Put $x = z$; $y = 0$.

* $F(z) = \int (V_y + i V_x)\, dz + c$

* $f(z) = \dfrac{F(z)}{1-i}$

① Find the analytic function $f(z) = P + iQ$

if $P - Q = \dfrac{\sin(2x)}{\cosh(2y) + \cos(2x)}$.

Sol

Take $U = \dfrac{\sin(2x)}{\cosh(2y) + \cos(2x)}$

$$U_y = -2\sin(2x)\left[\cosh(2y)+\cos(2x)\right]^{-2} \cdot \sinh(2y)$$

$$U_y = \frac{-2\sin(2x)\sinh(2y)}{\left[\cosh(2y)+\cos(2x)\right]^2}$$

Put $x = z$; $y = 0$

$$U_x = \frac{2\cos(2z)+2}{\left[1+\cos(2z)\right]^2} = \frac{2}{1+\cos(2z)}$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z$$

$$U_y = 0$$

$$F(z) = \int (U_x - iU_y)\, dz + c$$

$$= \int \sec^2 z \; dz + c$$

(2) Determine the analytic function $f(z) = u + iv$ given

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$ and

$f\left(\frac{\pi}{2}\right) = 0.$

Sol

Take $$U = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$$U_x = \frac{2(\cos x - \cosh y)\left[-\sin x + \cos x\right] + \left[\cos x + \sin x - e^{-y}\right] \cdot 2 \sin x}{4(\cos x - \cosh y)^2}$$

$$U_y = \frac{2(\cos x - \cosh y) \cdot e^{-y} + 2\left[\cos x + \sin x - e^{-y}\right] \sinh y}{4(\cos x) - \cosh y)^2}$$

Put $x = z \;\; ; \;\; y = 0.$

$$U_x = \frac{2(\cos z - 1)\left[-\sin z + \cos z\right] + 2\left[\cos z + \sin z - 1\right] \sin z}{4(\cos z - 1)^2}$$

$$= \frac{2(1-\cos z)}{4(\cos z-1)^2} = \frac{1}{2\times 2\sin^2\left(\frac{z}{2}\right)} = \frac{1}{4\sin^2\left(\frac{z}{2}\right)}$$

$$U_y = \frac{2(\cos z-1) + 2\left[\cos z+\sin z-1\right]\times 0}{4(\cos z-1)^2}$$

$$= \frac{2(\cos z-1)}{4(\cos z-1)^2} = \frac{-2(1-\cos z)}{4(1-\cos z)^2}$$

$$= \frac{-1}{2(1-\cos z)} = \frac{-1}{2\times 2\sin^2\left(\frac{z}{2}\right)}$$

$$U_y = \frac{-1}{4\sin^2\left(\frac{z}{2}\right)} = \frac{-1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$F(z) = \int (U_x - iU_y)\,dz + c'$$

$$= \int \left[\frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right) + \frac{i}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)\right] dz + c'$$

$$= \frac{1}{4}(1+i)\int \operatorname{cosec}^2\left(\frac{z}{2}\right) dz + c'$$

$$f(z) = \frac{F(z)}{1+i}$$

$$f(z) = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + \frac{c}{1+i} = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + C_1$$

Take $z = \pi/2$

$$0 = \frac{-1}{2}\cot\left(\frac{\pi}{4}\right) + C_1$$

$$= \frac{-1}{2} + C_1 \Rightarrow \boxed{C_1 = \frac{1}{2}}$$

$$\therefore \quad f(z) = \frac{-1}{2}\cot\left(\frac{z}{2}\right) + \frac{1}{2}.$$

(3) Find the analytic function $f(z) = u+iv$ if

$u+v = e^{x}\left[\cos y + \sin y\right]$

Sol

$$V = e^{x}\left[\cos y + \sin y\right]$$

$$V_x = e^{x}\left[\cos y + \sin y\right]$$

$$V_y = e^{x}\left[-\sin y + \cos y\right]$$

$= (1+i)\, e^{z} + c$

$\Rightarrow f(z) = \dfrac{F(z)}{1+i}$

$\Rightarrow f(z) = e^{z} + \dfrac{c}{1+i}$

Transformation

A complex valued function of complex variable $w = f(z)$ can be treated as a transformation of points of Z-plane into points of W-plane.

Invariant or Fixed point

The invariant (or) fixed points of the transformation $w = f(z)$ is given by solving the equation $z = f(z)$.

① Find the invariant points of z^2.

Sol

(2) Find the invariant points of the transformation $\omega = \frac{z-1}{z+1}$.

Sol

The invariant points are given by

$$z = \frac{z-1}{z+1}$$

$$\Rightarrow z(z+1) = z-1$$

$$\Rightarrow z^2+z = z-1$$

$$\Rightarrow z^2 = -1 \Rightarrow z = \pm i.$$

(3) Find the invariant points of $\omega = \frac{2z+6}{z+7}$.

Sol

The invariant points are given by

$$z = \frac{2z+6}{z+7}$$

$$\Rightarrow z^2+7z = 2z+6$$

4) Find the fixed points of the mapping $w = \frac{6z-9}{z}$

Sol

The fixed points are given by

$z = \frac{6z-9}{z} \Rightarrow z^2 = 6z-9$

$\Rightarrow z^2 - 6z + 9 = 0.$

$\Rightarrow (z-3)^2 = 0. \Rightarrow z = 3, 3.$

Bilinear Transformation

Any transformation of the form

$w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is called

Bilinear transformation.

Cross - Ratio

$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{\ldots}$

(1) Find the Bilinear transformation which maps the point $\infty, i, 0$ of z-plane into $0, i, \infty$ of the W-plane.

Sol

Z-plane	W-plane.
$z_1 = \infty$	$w_1 = 0$
$z_2 = i$	$w_2 = i$
$z_3 = 0$	$w_3 = \infty$.

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-\cancel{w_1})(w_2-\cancel{w_3})}{(\cancel{w-w_3})(w_2-w_1)} = \frac{(\cancel{z-z_1})(z_2-z_3)}{(z-z_3)(z_2-\cancel{z_1})}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow \frac{w}{i} = \frac{i}{z} \Rightarrow w = \frac{i^2}{z}$$

$$\Rightarrow \boxed{w = \frac{-1}{}}$$

z-plane	w-plane
$z_1 = 1$	$w_1 = 0$
$z_2 = i$	$w_2 = 1$
$z_3 = -1$	$w_3 = \infty$

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w - w_1)\cancel{(w_2 - w_3)}}{\cancel{(w - w_3)}(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{w - 0}{1 - 0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \quad w = \frac{-i(z-1)}{(z+1)} \Rightarrow \boxed{w = \frac{i - iz}{z+1}}$$

(3) Find the Bilinear transformation that

Z-plane W-plane

$z_1 = 0$ $\qquad w_1 = i$

$z_2 = -1$ $\qquad w_2 = 0$

$z_3 = i$ $\qquad w_3 = \infty$

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{\cancel{-}z(1+i)}{(z-i)\times\cancel{-}1}$$

$$\Rightarrow w-i = \frac{z}{z-i}\times\left[-i(1+i)\right]$$

$$\Rightarrow w-i = \frac{z(1-i)}{z-i}$$

$$\Rightarrow w = \frac{z(1-i)}{z-i} + i = \frac{z-\cancel{iz}+\cancel{iz}+1}{z-i}$$

④ Find the Bilinear transformation that maps the points $1, i, -1$ of Z-plane onto $i, 0, -i$ of W-plane.

Sol

Z-plane W-plane

$z_1 = 1$ $w_1 = i$

$z_2 = i$ $w_2 = 0$

$z_3 = -1$ $w_3 = -i$

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \frac{w-i}{w+i} = +i\frac{(z-1)}{(z+1)}$$

$$\Rightarrow \frac{w\cancel{-i} + w\cancel{+i}}{\cancel{w}+i - \cancel{w}+i} = \frac{iz - i + z + 1}{iz - i - z - 1}$$

$$\frac{\cancel{2}w}{-\cancel{2}i} = \frac{iz - i + z + 1}{iz - i - z - 1}$$

$$\Rightarrow \quad w = \frac{+z + 1 + iz + i}{iz - i - z - 1}$$

$$\Rightarrow \quad w = \frac{z(i+1) + (1+i)}{z(i-1) - (1+i)}$$

(5) Find the bilinear transformation that transforms the points $z = 1, i, -1$ of the Z-plane into the points $w = 2, i, -2$ of the W-plane.

Sol

Z-plane W-plane

$Z_1 = 1$ $\qquad\qquad$ $w_1 = 2$

$Z_2 = i$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)\ (i+2)}{(w+2)\ (i-2)} = \frac{(z-1)\ (i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} \times \left(\frac{-3}{5} - \frac{4}{5}i\right) = \frac{z-1}{z+1} \times -i$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1} \times \frac{-i}{\left(\frac{-3}{5} - \frac{4}{5}i\right)}$$

$$\Rightarrow \frac{w-2}{w+2} = \left(\frac{4}{5} + \frac{3}{5}i\right) \frac{z-1}{z+1}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{(a+bi)(z-1)}{z+1}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{az - a + biz - bi}{z+1}$$

$$\therefore\ w = -2 \times \frac{\left[\frac{9}{5}z+\frac{1}{5}\right]+i\,(z-1)\frac{3}{5}}{\left[\frac{-1}{5}z-\frac{9}{5}\right]+i\,(z-1)\frac{3}{5}}$$

$$\Rightarrow\ w = -2 \times \frac{(9z+1)+i\,(3z-3)}{(-z-9)+i\,(3z-3)}$$

⑥ Find the bilinear transformation which maps the points $-i, 0, i$ into the points, $-1, i, 1$ respectively. Into what curve, the Y-axis is transformed under this transformation?

Sol

Z-plane	W-plane
$z_1 = -i$	$w_1 = -1$
$z_2 = 0$	$w_2 = i$
$z_3 = i$	$w_3 = 1$

$$\frac{(\omega+1)\,(i-1)}{(\omega-1)\,(i+1)} = \frac{(z+i)\,(0-i)}{(z-i)\,(0+i)}$$

$$\frac{(\omega+1)}{(\omega-1)} = -\frac{(z+i)}{(z-i)} \times \frac{(i+1)}{(i-1)}$$

$$\frac{\omega+1}{\omega-1} = \frac{iz-1}{z-i}$$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\frac{\omega+\cancel{1}+\omega-\cancel{1}}{\cancel{\omega}+1-\cancel{\omega}+1} = \frac{iz-1+zi-i}{iz-1-z+i}$$

$$\frac{\cancel{2}\omega}{\cancel{2}} = \frac{(i+1)z-(1+i)}{(i-1)z-(1-i)}$$

$$\boxed{\omega = \frac{(i+1)z-(1+i)}{(i-1)z-(1-i)}}$$

Critical points

Consider $\omega = f(z)$.

The critical points are the points at which

$$\frac{dw}{dz} = 0 \quad \text{and} \quad \frac{dz}{dw} = 0.$$

① Find the critical points of $\omega^2 = (z-\alpha)(z-\beta)$.

Sol

$$\omega^2 = (z-\alpha)(z-\beta).$$

$$2\omega \cdot \frac{dw}{dz} = (z-\alpha) + (z-\beta) = 2z-\alpha-\beta$$

$$\frac{dw}{dz} = \frac{2z-\alpha-\beta}{2\omega} \quad \& \quad \frac{dz}{dw} = \frac{2\omega}{2z-\alpha-\beta}$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z-\alpha-\beta = 0 \Rightarrow z = \frac{\alpha+\beta}{2}$$

$$\frac{dz}{dw} = 0 \Rightarrow 2\omega = 0 \Rightarrow \omega = 0 \Rightarrow \omega^2 = 0$$

$$\Rightarrow (z-\alpha)(z-\beta) = 0.$$

$$\Rightarrow z = \alpha, \beta$$

(2) Find the critical points of $\omega = z^2$.

Sol

$$\omega = z^2$$

$$\frac{dw}{dz} = 2z \qquad \frac{dz}{dw} = \frac{1}{2z}$$

$$\frac{dw}{dz} = 0 \Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$$

$$\frac{dz}{dw} = 0 \Rightarrow \frac{1}{2z} = 0 \Rightarrow \boxed{z = \infty}$$

Conformal Mapping

A mapping $\omega = f(z)$ that preserves angle between ~~any~~ every pair of curves both in magnitude and direction is called conformal mapping.

Type (1) : ~~$w = z + c$~~

Sol

$$\omega = 1 + \frac{2}{z} = 1 + 2z^{-1}$$

$$\frac{d\omega}{dz} = -2z^{-2} = \frac{-2}{z^2} \qquad \frac{dz}{d\omega} = \frac{z^2}{-2}$$

The critical points are given by

$$\frac{d\omega}{dz} = 0 \Rightarrow \frac{-2}{z^2} = 0 \Rightarrow \boxed{z = \infty}$$

$$\frac{dz}{d\omega} = 0 \Rightarrow \frac{z^2}{-2} = 0 \Rightarrow \boxed{z = 0.}$$

Type (i) : $\omega = c + z$

(1) Find the image of $2x + y - 3 = 0$ under $\omega = z + 2i$.

Sol

$$\omega = z + 2i$$

$$u + iv = x + iy + 2i$$

$$u + iv = x + i(2 + y)$$

(2) Find the image of $|z| = 2$ under $w = z + 3 + 2i$.

Sol

$$w = z + 3 + 2i$$

$$u + iv = x + iy + 3 + 2i$$

$$u + iv = x + 3 + i(y + 2)$$

$u = x + 3$ $\qquad$ $v = y + 2$

$x = u - 3$ $\qquad$ $y = v - 2$

The image of $|z| = 2$ is

$|z| = 2 \Rightarrow |z|^2 = 4 \Rightarrow x^2 + y^2 = 4$

$\Rightarrow (u-3)^2 + (v-2)^2 = 4.$

Type (2) : $w = cz$

(1) Find the map of the circle $|z| = 3$ under the transformation $w = 2z$

Sol

$$w = 2z$$

The image of $|z| = 3$ is

$|z| = 3$

$\Rightarrow |z|^2 = 9$

$\Rightarrow x^2 + y^2 = 9$

$\Rightarrow \frac{u^2}{4} + \frac{v^2}{4} = 9 \Rightarrow u^2 + v^2 = 36.$

(2) Find the image of $0 \le x \le 2$ under $w = iz$.

Sol

$w = iz$

$\Rightarrow u + iv = i(x + iy)$

$\Rightarrow u + iv = ix - y$

$\Rightarrow u = -y \qquad v = x$

$\Rightarrow \boxed{y = -u} \qquad \boxed{x = v}$

Given $0 \le x \le 2$

$\Rightarrow 0 \le v \le 2$

Type (3) $W = e^z$

$\Rightarrow \log\rho + \log e^{i\varphi} = x+iy$

$\Rightarrow \log\rho + i\varphi = x+iy$

From this, $x = \log\rho \Rightarrow x = \frac{1}{2}\log(u^2+v^2)$

$y = \varphi \Rightarrow y = \tan^{-1}\left(\frac{v}{u}\right)$.

(1) Find the image of the straight line $y=x$ under $w=e^z$.

Sol

$w = e^z \Rightarrow x = \log\rho \; ; \; y = \varphi$

$y = x \Rightarrow \cancel{\varphi = \log\rho}$

$\Rightarrow \cancel{\tan^{-1}\left(\frac{v}{u}\right) = \ldots} \; \tan^{-1}\left(\frac{v}{u}\right) = \frac{1}{2}\log(u^2+v^2)$.

Type (4) – $w = z^2$

$w = z^2 \Rightarrow z = w^{1/2}$

$\Rightarrow x+iy = \left(\rho e^{i\varphi}\right)^{1/2} = \rho^{1/2} \cdot e^{i\varphi/2}$

$\Rightarrow x+iy = \rho^{1/2}\left[\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right]$

Cardioid $\rho = 2(1+\cos\varphi)$

Sol

$$w = z^2 \Rightarrow x = \rho^{1/2} \cos\left(\frac{\varphi}{2}\right)$$

$$y = \rho^{1/2} \sin\left(\varphi/2\right).$$

$$|z-1| = 1 \Rightarrow |x+iy-1| = 1$$

$$\Rightarrow |(x-1)+iy| = 1$$

$$\Rightarrow (x-1)^2 + y^2 = 1$$

$$\Rightarrow x^2 - 2x + 1 + y^2 = 1$$

$$\Rightarrow x^2 - 2x + y^2 = 0.$$

$$\Rightarrow \rho - 2\rho^{1/2} \cos\left(\frac{\varphi}{2}\right) = 0$$

$$\Rightarrow \rho = 2\rho^{1/2} \cos\left(\frac{\varphi}{2}\right)$$

$$\Rightarrow \rho^2 = 4\rho \cos^2\left(\frac{\varphi}{2}\right)$$

$$\Rightarrow \rho = 4\cos^2\left(\frac{\varphi}{2}\right) = 2\left\{2\cos^2\left(\frac{\varphi}{2}\right)\right\}$$

$\Rightarrow$

Type (5) :- $\quad w = \frac{1}{z}$

$$\cancel{u+iv} = \cancel{\frac{1}{x+iy}} \times \cancel{\frac{x-iy}{x-iy}}$$

$$\cancel{u+iv} \quad u+iv = \frac{1}{x+iy}$$

$$\Rightarrow \quad x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow \quad x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \quad x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

① Find the image of the half plane $x > c$, $c > 0$ under the transformation $w = \frac{1}{z}$. Show the regions graphically.

Sol

$$w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x+iy = \frac{1}{\quad} \times \frac{u-iv}{\quad}$$

$$x = \frac{u}{u^2+v^2} \quad ; \quad y = \frac{-v}{u^2+v^2}$$

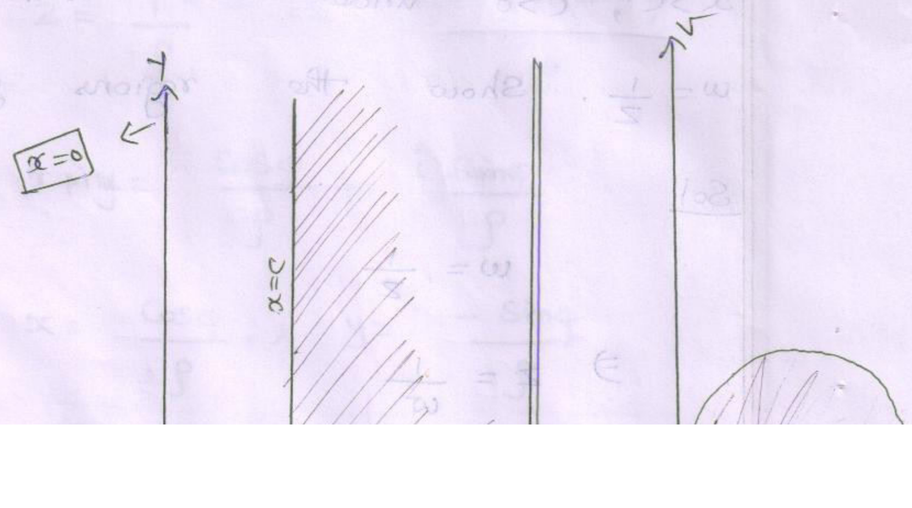
$$x > c \Rightarrow \frac{u}{u^2+v^2} > c$$

$$\Rightarrow \frac{u}{c} > u^2+v^2$$

$$\Rightarrow u^2+v^2-\frac{u}{c} < 0.$$

$$\Rightarrow u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2 + v^2 < \left(\frac{1}{2c}\right)^2$$

$$\Rightarrow \left(u-\frac{1}{2c}\right)^2 + v^2 < \left(\frac{1}{2c}\right)^2 \Rightarrow$$ Inside of the circle whose center is $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2c}$.



(8) Find the image of $|z+1|=1$ under the transformation $w = \frac{1}{z}$

Sol

$$w = \frac{1}{z}$$

$$\Rightarrow \quad z = \frac{1}{w}$$

$$\Rightarrow \quad x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow \quad x+iy = \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \quad x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

$$|z+1| = 1 \Rightarrow |x+iy+1| = 1$$

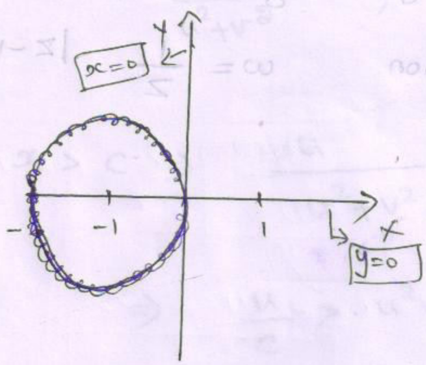
$$\Rightarrow |(x+1)+iy| = 1$$

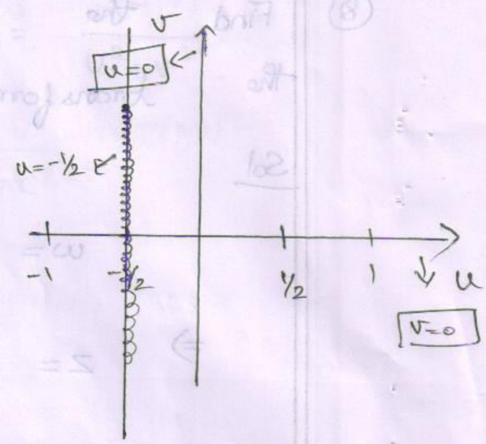
$$\Rightarrow (x+1)^2 + y^2 = 1 \rightarrow$$ Circle with center $(-1,0)$ & radius 1.

$$\Rightarrow x^2 + 2x + 1 + y^2 = 1$$

$$\Rightarrow x^2 + 2x + y^2 = 0.$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{2u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 0.$$





③ Find the image of the hyperbola $x^2-y^2=1$ under the transformation $\omega=\frac{1}{z}$

Sol

$$\omega=\frac{1}{z}$$

$$\Rightarrow \quad z=\frac{1}{\omega}=\frac{1}{\rho e^{i\varphi}}$$

$$\Rightarrow \quad z=\frac{1}{\rho}e^{-i\varphi}=\frac{\cos\varphi-i\sin\varphi}{\rho}$$

$$\Rightarrow \quad x+iy=\frac{\cos\varphi}{\rho}-i\frac{\sin\varphi}{\rho}$$

$$\Rightarrow \quad x=\frac{\cos\varphi}{\rho}\ , \quad y=\frac{-\sin\varphi}{\rho}$$

(4) Find the image in the W-plane of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $\omega = \frac{1}{z}$.

Sol

$$\omega = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{\omega}$$

$$\Rightarrow x+iy = \frac{1}{u+iv}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad ; \quad y = \frac{-v}{u^2+v^2}$$

$$\frac{1}{4} \le y \le \frac{1}{2}$$

$\frac{1}{4} \le y$

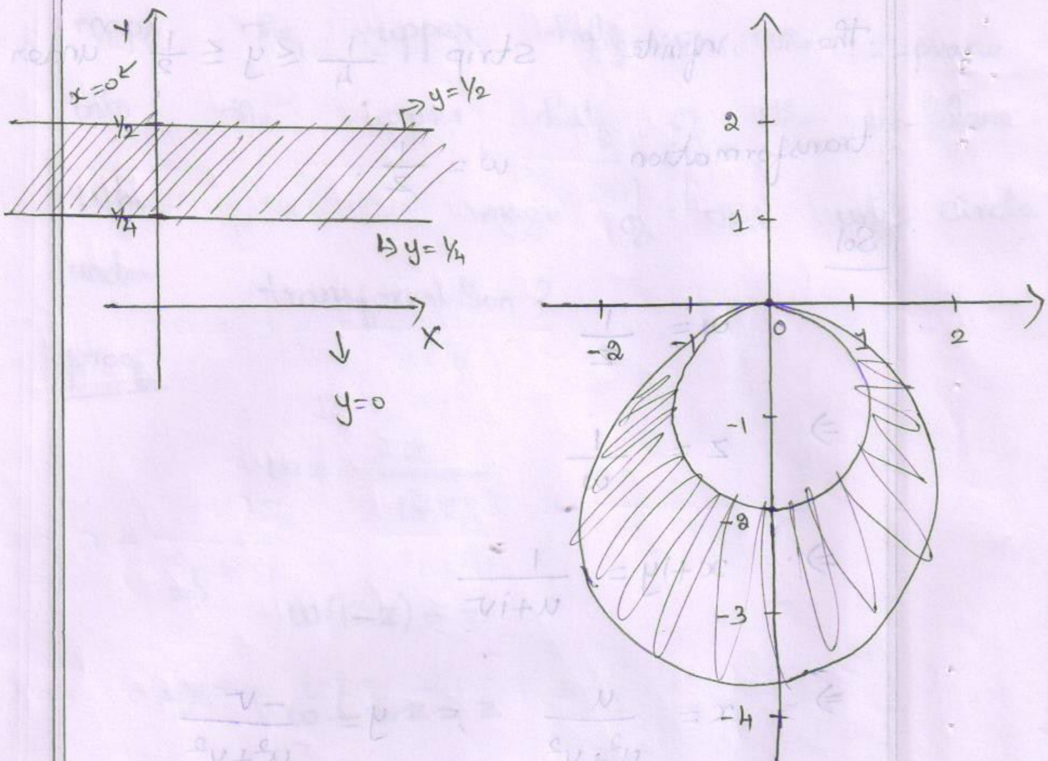
$\frac{1}{4} \le \frac{-v}{u^2+v^2}$

$\Rightarrow u^2+v^2 \le -4v$

$y \le \frac{1}{2}$

$\frac{-v}{u^2+v^2} \le \frac{1}{2}$

$u^2+v^2 \ge -2v$



⑤ Show that the transformation $\omega = \frac{1}{z}$ transforms all circles and straight lines in the z-plane into circles or straight lines in the w-plane.

Sol

$$\omega = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{\omega} \Rightarrow x + iy = \frac{1}{u + iv}$$

(1) represents a circle when $a \neq 0$.

(1) represents a st. line when $a = 0$.

$$(1) \Rightarrow a\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0.$$

$$\Rightarrow \frac{a}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0.$$

$$\Rightarrow c(u^2+v^2) + 2gu - 2fv + a = 0 \quad \text{— (2)}$$

(2) represents a circle when $c \neq 0$.

(2) represents a st. line when $c = 0$.

I : $a \neq 0$; $c \neq 0$.

Circle is mapped onto a circle.

II $a \neq 0$, $c = 0$

Circle is mapped onto st. line

III $a = 0$, $c \neq 0$.

(6) Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane. What is the image of the unit circle under transformation?

<u>proof</u>

$$w = \frac{z}{1-z}$$

$$\Rightarrow \quad w(1-z) = z$$

$$\Rightarrow \quad w - wz = z$$

$$\Rightarrow \quad z + wz = w$$

$$\Rightarrow \quad z(1+w) = w$$

$$\Rightarrow \quad z = \frac{w}{1+w}$$

$$\Rightarrow \quad x+iy = \frac{[u+iv]}{[(1+u)+iv]} \times \frac{[(1+u)-iv]}{[(1+u)-iv]}$$

$$= \quad u(1+u)+v^2 + i\left[(1+u)v - uv\right]$$

Upper half of z-plane : $y > 0$

$$\Rightarrow \frac{v}{(1+u)^2 + v^2} > 0$$

$$\Rightarrow v > 0$$

↳ Upper half of w-plane.

Unit circle : $|z| = 1$

$$\left|\frac{w}{1+w}\right| = 1$$

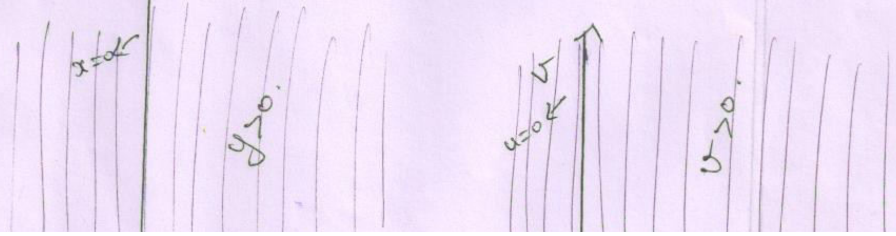
$$\Rightarrow |w| = |1+w|$$

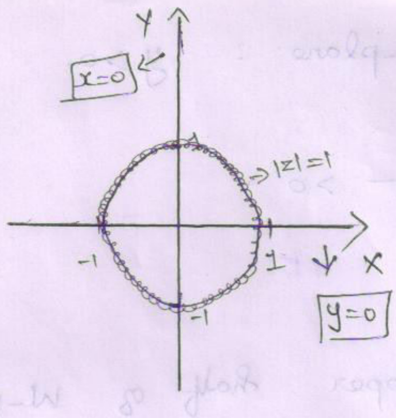
$$\Rightarrow |u+iv| = |(1+u)+iv|$$

$$\Rightarrow u^2+v^2 = (1+u)^2+v^2$$

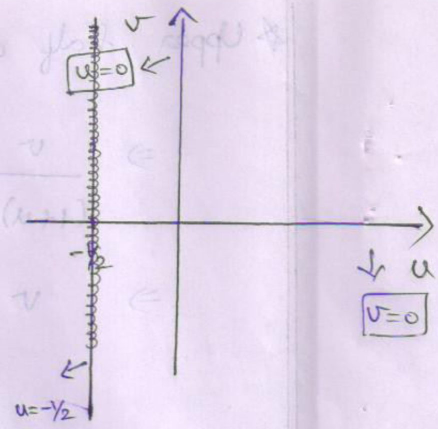
$$\Rightarrow u^2+v^2 = 1+u^2+2u+v^2$$

$$\Rightarrow 2u+1=0 \Rightarrow \boxed{u = \frac{-1}{2}}$$





y
x=0
|z|=1
-1
1
-1
x
y=0



v
u=0
-1/2
u
v=0
u=-1/2